Noncommutative Spheres and Instantons

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Abstract

We report on some recent work on deformation of spaces, notably deformation of spheres, describing two classes of examples.

The first class of examples consists of noncommutative manifolds associated with the so called θ-deformations which were introduced in [17] out of a simple analysis in terms of cycles in the (b, B)-complex of cyclic homology. These examples have non-trivial global features and can be endowed with a structure of noncommutative manifolds, in terms of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. In particular, noncommutative spheres S_θ^N are isospectral deformations of usual spherical geometries. For the corresponding spectral triple $(C^\infty(S_\theta^N), \mathcal{H}, D)$, both the Hilbert space of spinors $\mathcal{H} = L^2(S^N, \mathcal{S})$ and the Dirac operator D are the usual ones on the commutative N-dimensional sphere S^N and only the algebra and its action on $\mathcal{H}$ are deformed. The second class of examples is made of the so called quantum spheres S_q^N which are homogeneous spaces of quantum orthogonal and quantum unitary groups. For these spheres, there is a complete description of K-theory, in terms of nontrivial self-adjoint idempotents (projections) and unitaries, and of the K-homology, in term of nontrivial Fredholm modules, as well as of the corresponding Chern characters in cyclic homology and cohomology.

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Contents

1 Introduction

We shall describe two classes of deformation of spaces with particular emphasis on spheres.

The first class of examples are noncommutative manifolds associated with the so called θ-deformations and which are constructed naturally [17] by a simple analysis in terms of cycles in the (b, B)-complex of cyclic homology. These examples have non-trivial global features and can be endowed with a structure of noncommutative manifolds, in terms of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ [10, 12]. In particular we shall describe noncommutative spheres S_θ^N which are isospectral deformations of usual spherical geometries; and we shall also show quite generally that any compact Riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries.

The second class of examples is made of the so called quantum orthogonal spheres S_q^N, which have been constructed as homogeneous spaces [30] of quantum orthogonal groups $\mathrm{SO}_q(N+1)$ and quantum unitary spheres S_q^{2n+1} which are homogeneous spaces of quantum unitary groups $\mathrm{SU}_q(n+1)$ [59]. The quantum groups $\mathrm{SO}_q(N+1)$ and $\mathrm{SU}_q(n+1)$ are R-matrix deformations of the usual orthogonal and unitary groups $\mathrm{SO}(N+1)$ and $\mathrm{SU}_q(n+1)$ respectively. In fact, it has been remarked in [34] that 'odd' quantum orthogonal spheres are the same as 'odd' quantum unitary ones, as it happens for undeformed spheres.

It is not yet clear if and to which extend these quantum spheres can be endowed with the structure of a noncommutative geometry via a spectral triple. There has been some interesting work in this direction recently. In [6] a 3-summable spectral triple was constructed for $\mathrm{SU}_q(2)$; this has been thoroughly analyzed in [14] in the context of the noncommutative local index formula of [18]. A 2-summable spectral triple on $\mathrm{SU}_q(2)$ was constructed in [7] together with a spectral triple on the spheres S_{qc}^2 of Podleś [49]. Also, a '0-summable' spectral triple on the so called standard spheres S_{q0}^2 has been given in [25, 55, 38]. Instead, on these spheres one can construct Fredholm modules, which provide a structure which is somewhat weaker that the one given by spectral triples. Indeed, a Fredholm module can be though of as a noncommutative conformal structure [20]. This construction for the quantum spheres S_q^N will be described in Section 6 closely following the paper [34].

All our spaces can be regard as "noncommutative real affine varieties". For such an object, X, the algebra $A(X)$ is a finitely presented $*$-algebra in terms of generators and relations. In contrast with classical algebraic geometry, there does not in general exist a topological point set X. Nevertheless, we regard X as a noncommutative space and $A(X)$ as the algebra of polynomial functions on X. In the classical case, one can consider the algebra of continuous functions on the underlying topological space of an affine variety. If X is bounded, then this is a C*-algebra and is the completion of $A(X)$. In general, one defines $\mathcal{C}(X)$ to be the C*-algebraic completion of the $*$-algebra $A(X)$. To construct this, one first considers the free algebra $F(X)$ on the same generators of the algebra $A(X)$. Then, one takes all possible $*$-representations π of $F(X)$ as bounded operators on a countably infinite-dimensional Hilbert space $\mathcal{H}$. The representations are taken to be *admissible*, that is in $\mathcal{B}(\mathcal{H})$ the images of the generators of $F(X)$ satisfy the same defining relations as in $A(X)$. For $a \in F(X)$ one defines $\|a\| = \mathrm{Sup} \|\pi(a)\|$ with π ranging through all admissible representations of $F(X)$. It turns out that $\|a\|$ is finite for $a \in F(X)$ and $\|\cdot\|$ is a seminorm. Then $\mathcal{I} := \{a \in F(X) = 0\}$ is a two-sided ideal and one obtains a C*-norm on $F(X)/\mathcal{I}$. The C*-algebra $\mathcal{C}(X)$ is the completion of $F(X)/\mathcal{I}$ with respect to

this norm. The C*-algebra $\mathcal{C}(X)$ has the universal property that any *-morphism from $A(X)$ to a separable C*-algebra factors through $\mathcal{C}(X)$. In particular, any *-representation of $A(X)$ extends to a representation of $\mathcal{C}(X)$.

The word instanton in the title refers to the fact that all (in particular even) spheres come equipped with a projection $e \in \mathrm{Mat}_r(A(X)), e^2 = e = e^*$, for $X = S_\theta^N$ and $X = S_q^N$. These projection determines the module of sections of a vector bundle which deforms the usual monopole bundle and instanton bundle in two and four dimensions respectively, and generalizes them in all dimensions.

In particular on the four dimensional S_θ, one can develop Yang-Mills theory, since there are all the required structures, namely the algebra, the calculus and the "vector bundle" e (naturally endowed, in addition, with a preferred connection ∇). Among other things there is a basic inequality showing that the Yang-Mills action, $YM(\nabla) = \fint \theta^2 \, ds^4$, (where $\theta = \nabla^2$ is the curvature, and $ds = D^{-1}$) has a strictly positive lower bound given by the topological invariant $\varphi(e) = \fint \gamma (e - \frac{1}{2})[D, e]^4 \, ds^4$ which, for the canonical projections turns out to be just 1: $\varphi(e) = 1$.

In general, the projection e for the spheres S_θ^{2n} satisfies self-duality equations

$$*_H e(de)^n = \mathrm{i}^n e(de)^n, \tag{1.1}$$

with a suitably defined Hodge operator $*_H$ [15] (see also [1] and [41]).

An important problem is the construction and the classification of Yang-Mills connections in the noncommutative situation along the line of the ADHM construction [3]. This was done in [19] for the noncommutative torus and in [48] for a noncommutative $\mathbb{R}^4$.

It is not yet clear if a construction of gauge theories along similar lines can be done for the quantum spheres $X = S_q^N$.

There has been recently an explosion of work on deformed spheres from many points of view. The best I can do here is to refer to [22] for an overview of noncommutative and quantum spheres in dimensions up to four. In [58] there is a family of noncommutative 4-spheres which satisfy the Chern character conditions of [17] up to cohomology classes (and not just representatives). Additional quantum 4-dimensional spheres together with a construction of quantum instantons on them is in [32]. A different class of spheres in any even dimension was proposed in [4]. At this workshop T. Natsume presented an example in two dimensions [47].

2   Instanton Algebras

In this Section we shall describe how to obtain in a natural way noncommutative spaces (i.e. algebras) out of the Chern characters of idempotents and unitaries in cyclic homology. For this we shall give a brief overview of the needed fundamentals of the theory, following [5]. For later use we shall also describe the dual cohomological theories.

2.1 Hochschild and cyclic homology and cohomology

Given an algebra $\mathcal{A}$, consider the chain complex $(C_*(\mathcal{A}) = \bigoplus_n C_n(\mathcal{A}), b)$ with $C_n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}$ and the boundary map b defined by

$$b \quad : \quad C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A}) \,,$$

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \quad := \quad \sum_{j=0}^{n-1}(-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$

$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \,. \tag{2.1}$$

It is easy to prove that $b^2 = 0$. The *Hochschild homology* $HH_*(\mathcal{A})$ of the algebra $\mathcal{A}$ is the homology of this complex,

$$HH_n(\mathcal{A}) := H_n(C_*(\mathcal{A}), b) = Z_n/B_n \,, \tag{2.2}$$

with the cycles given by $Z_n := \ker(b : C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A}))$ and the boundary given by $B_n := \mathrm{im}(b : C_{n+1}(\mathcal{A}) \to C_n(\mathcal{A}))$. We have another operator which increases the degree

$$B : C_n(\mathcal{A}) \to C_{n+1}(\mathcal{A}) \,, \qquad B = B_0 A \,, \tag{2.3}$$

where

$$B_0(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \mathbb{I} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \tag{2.4}$$

$$A(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \frac{1}{n+1} \sum_{j=0}^{n}(-1)^{nj} a_j \otimes a_{j+1} \otimes \cdots \otimes a_{j-1} \,, \tag{2.5}$$

with the obvious cyclic identification $n + 1 = 0$. Again it is straightforward to check that $B^2 = 0$ and that $bB + Bb = 0$.

By putting together these two operators, one gets a bi-complex $(C_*(\mathcal{A}), b, B)$ with $C_{p-q}(\mathcal{A})$ in bi-degree p, q. The *cyclic homology* $HC_*(\mathcal{A})$ of the algebra $\mathcal{A}$ is the homology of the total complex $(CC(\mathcal{A}), b + B)$, whose n-th term is given by $CC_n(\mathcal{A}) := \bigoplus_{p+q=n} C_{p-q}(\mathcal{A}) = \bigoplus_{0 \le q \le [n/2]} C_{2n-q}(\mathcal{A})$. This bi-complex may be best organized in a plane diagram whose vertical arrows are associated with the operator b and whose horizontal ones are associated with the operator B,

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \\
b \downarrow & & b \downarrow & & b \downarrow \\
C_2(\mathcal{A}) & \xleftarrow{B} & C_1(\mathcal{A}) & \xleftarrow{B} & C_0(\mathcal{A}) \\
b \downarrow & & b \downarrow & & \\
C_1(\mathcal{A}) & \xleftarrow{B} & C_0(\mathcal{A}) & & \\
b \downarrow & & & & \\
C_0(\mathcal{A}) & & & &
\end{array}
\tag{2.6}
$$

The n-th term $CC_n(\mathcal{A})$ of the total complex is just the n-th (NW – SE) diagonal in the diagram (2.6). Then,

$$HC_n(\mathcal{A}) := H_n(CC(\mathcal{A}), b + B) = Z_n^\lambda / B_n^\lambda , \qquad (2.7)$$

with the *cyclic cycles* given by $Z_n^\lambda := \ker(b + B : CC_n(\mathcal{A}) \to CC_{n-1}(\mathcal{A}))$ and the *cyclic boundaries* given by $B_n^\lambda := \operatorname{im}(b + B : CC_{n+1}(\mathcal{A}) \to CC_n(\mathcal{A}))$.

Example 1 *If M is a compact manifold, the Hochschild homology of the algebra of smooth functions $C^\infty(M)$ gives the de Rham complex (Hochschild-Konstant-Rosenberg theorem),*

$$\Omega_{dR}^k(M) \simeq HH_k(C^\infty(M)) , \qquad (2.8)$$

with $\Omega_{dR}^k(M)$ the space of de Rham forms of order k on M. If d denotes the de Rham exterior differential, this isomorphisms is implemented by

$$a_o da_1 \wedge \cdots \wedge da_k \mapsto \varepsilon_k(a_0 \otimes a_1 \otimes \cdots \otimes da_k) \qquad (2.9)$$

where ε_k is the antisymmetrization map

$$\varepsilon_k(a_0 \otimes a_1 \otimes \cdots \otimes da_k) := \sum_{\sigma \in S_k} sign(\sigma)(a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes da_{\sigma(k)}) \qquad (2.10)$$

and S_k is the symmetric group of order k. In particular one checks that $b \circ \varepsilon_k = 0$. The de Rham differential d corresponds to the operator B_* (the lift of B to homology) in the sense that

$$\varepsilon_{k+1} \circ d = (k+1) B_* \circ \varepsilon_k . \qquad (2.11)$$

On the other hand, the cyclic homology gives [9, 44]

$$HC_k(C^\infty(M)) = \Omega_{dR}^k(M)/d\Omega_{dR}^{k-1}(M) \oplus H_{dR}^{k-2}(M) \oplus H_{dR}^{k-4}(M) \oplus \cdots , \qquad (2.12)$$

where $H_{dR}^j(M)$ is the j-th de Rham cohomology group. The last term in the sum is $H_{dR}^0(M)$ or $H_{dR}^1(M)$ according to wether k is even or odd.
From the fact that $C^\infty(M)$ is commutative it follows that there is a natural decomposition (the λ-decomposition) of cyclic homology in smaller pieces,

$$HC_0(C^\infty(M)) = HC_0^{(0)}(C^\infty(M)) ,$$
$$HC_k(C^\infty(M)) = HC_k^{(k)}(C^\infty(M)) \cdots \oplus HC_k^{(1)}(C^\infty(M)) , \qquad (2.13)$$

which is obtained by suitable idempotents $e_k^{(i)}$ which commute with the operator B: $Be_k^{(i)} = e_{k+1}^{(i+1)}B$. The previous decomposition corresponds to the decomposition in (2.12) and give a way to extract the de Rham cohomology

$$HC_k^{(k)}(C^\infty(M)) = \Omega_{dR}^k(M)/d\Omega_{dR}^{k-1}(M) ,$$
$$HC_k^{(i)}(C^\infty(M)) = H_{dR}^{2i-k}(M) , \qquad \text{for} \quad [n/2] \le i < n , \qquad (2.14)$$
$$HC_k^{(i)}(C^\infty(M)) = 0 , \qquad \text{for} \quad i < [n/2] ,$$

Looking at this example, one may think of cyclic homology as a generalization of de Rham cohomology to the noncommutative setting.

A Hochschild k cochain on the algebra $\mathcal{A}$ is an $(n+1)$-linear functional on $\mathcal{A}$ or a linear form on $\mathcal{A}^{\otimes(n+1)}$. Let $C^n(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C})$ be the collection of such cochains. We have a cochain complex $(C^*(\mathcal{A}) = \bigoplus_n C^n(\mathcal{A}), b)$ with a coboundary map, again denoted with the symbol b, defined by

$$b \; : \; C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}) \; ,$$

$$b\varphi(a_0, a_1, \cdots, a_{n+1}) \; := \; \sum_{j=0}^{n}(-1)^j\varphi(a_0, \cdots, a_j a_{j+1}, \cdots, a_{n+1})$$

$$+ (-1)^{n+1}\varphi(a_{n+1}a_0a_1, \cdots, a_n) \, . \tag{2.15}$$

Clearly $b^2 = 0$ and the *Hochschild cohomology* $HH^*(\mathcal{A})$ of the algebra $\mathcal{A}$ is the cohomology of this complex,

$$HH^n(\mathcal{A}) := H^n(C^*(\mathcal{A}), b) = Z^n/B^n \; , \tag{2.16}$$

with the cocycles given by $Z^n := \ker(b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}))$ and the coboundaries given by $B^n := \mathrm{im}(b : C^{n-1}(\mathcal{A}) \to C^n(\mathcal{A}))$.

A Hochschild 0-cocycle τ on the algebra $\mathcal{A}$ is a *trace*, since $\tau \in \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ and the cocycle condition is

$$\tau(a_0a_1) - \tau(a_1a_0) = b\tau(a_0, a_1) = 0 \; . \tag{2.17}$$

The trace property is extended to higher orders by saying that an n-cochain φ is *cyclic* if $\lambda\varphi = \varphi$, with

$$\lambda\varphi(a_0, a_1, \cdots, a_n) = (-1)^n\varphi(a_n, a_0, \cdots, a_{n-1}) \; . \tag{2.18}$$

A *cyclic cocycle* is a cyclic cochain φ for which $b\varphi = 0$.

A straightforward computation shows that the sets of cyclic n-cochains $C^n_\lambda(\mathcal{A}) = \{\varphi \in C^n(\mathcal{A}) \, , \, \lambda\varphi = \varphi\}$ are preserved by the Hochschild boundary operator: $(1 - \lambda)\varphi = 0$ implies that $(1 - \lambda)b\varphi = 0$. Thus we get a subcomplex $(C^*_\lambda(\mathcal{A}) = \bigoplus_n C^n_\lambda(\mathcal{A}), b)$ of the complex $(C^*(\mathcal{A}) = \bigoplus_n C^n(\mathcal{A}), b)$. The *cyclic cohomology* $HC^*(\mathcal{A})$ of the algebra $\mathcal{A}$ is the cohomology of this subcomplex,

$$HC^n(\mathcal{A}) := H^n(C^*_\lambda(\mathcal{A}), b) = Z^n_\lambda/B^n_\lambda \; , \tag{2.19}$$

with the cyclic cocycles given by $Z^n_\lambda := \ker(b : C^n_\lambda(\mathcal{A}) \to C^{n+1}_\lambda(\mathcal{A}))$ and the cyclic coboundaries given by $B^n := \mathrm{im}(b : C^{n-1}_\lambda(\mathcal{A}) \to C^n_\lambda(\mathcal{A}))$.

One can also define an operator B which is dual to the one in (2.3) for the homology and give a bicomplex description of cyclic cohomology by giving a diagram dual to the one in (2.6) with all arrows inverted and all indices 'up'. Since we shall not need this description later on, we only refer to [10] for all details. We mention an additional important operator, the *periodicity operator* S which is a map of degree 2 between cyclic cocycle,

$$S : Z^{n-1}_\lambda \longrightarrow Z^{n+1}_\lambda \; , \tag{2.20}$$

$$S\varphi(a_0, a_1, \cdots, a_{n+1}) := -\frac{1}{n(n+1)}\sum_{j=1}^{n}\varphi(a_0, \cdots, a_{j-1}a_ja_{j+1}, \cdots, a_{n+1})$$

$$-\frac{1}{n(n+1)}\sum_{1\leq i<j\leq n}(-1)^{i+j}\varphi(a_0, \cdots, a_{i-1}a_i, \cdots, a_ja_{j+1}, \cdots, a_{n+1}) \; .$$

One shows that $S(Z_\lambda^{n-1}) \subseteq Z_\lambda^{n+1}$. In fact $S(Z_\lambda^{n-1}) \subseteq B^{n+1}$, the latter being the Hochschild coboundary; and cyclicity is easy to show.

The induced morphisms in cohomology $S : HC^n \to HC^{n+2}$ define two directed systems of abelian groups. Their inductive limits

$$HP^0(\mathcal{A}) := \varinjlim HC^{2n}(\mathcal{A}) , \quad HP^1(\mathcal{A}) := \varinjlim HC^{2n+1}(\mathcal{A}) , \tag{2.21}$$

form a $\mathbb{Z}_2$-graded group which is called the *periodic cyclic cohomology* $HP^*(\mathcal{A})$ of the algebra $\mathcal{A}$.

'Il va sans dire': there is also a *periodic cyclic homology* [10, 44].

2.2 Noncommutative algebras from idempotents

Let $\mathcal{A}$ be an algebra (over $\mathbb{C}$) and let $e \in \text{Mat}_r(\mathcal{A})$, $e^2 = e$, be an idempotent. Its *even* (reduced) Chern character is a formal sum of chains

$$\text{ch}_*(e) = \sum_k \text{ch}_k(e) , \tag{2.22}$$

with the component $\text{ch}_k(e)$ an element of $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes 2k}$, where $\overline{\mathcal{A}} = \mathcal{A}/\mathbb{C}1$ is the quotient of $\mathcal{A}$ by the scalar multiples of the unit 1. The formula for $\text{ch}_k(e)$ is (with λ_k a normalization constant),

$$\begin{aligned}
\text{ch}_k(e) &= \left\langle \left(e - \frac{1}{2} \mathbb{I}_r \right) \otimes e \otimes e \cdots \otimes e \right\rangle \\
&= \lambda_k \sum \left(e_{i_0 i_1} - \frac{1}{2} \delta_{i_0 i_1} \right) \otimes \widetilde{e}_{i_1 i_2} \otimes \widetilde{e}_{i_2 i_3} \cdots \otimes \widetilde{e}_{i_{2k} i_0}
\end{aligned} \tag{2.23}$$

where δ_{ij} is the usual Kronecker symbol and only the class $\widetilde{e}_{i_j i_{j+1}} \in \overline{\mathcal{A}}$ is used in the formula. The crucial property of the character $\text{ch}_*(e)$ is that it defines a cycle [9, 10, 44, 13] in the reduced (b, B)-bicomplex of cyclic homology described above,

$$(b + B) \text{ch}_*(e) = 0 , \quad B \text{ch}_k(e) = b \text{ch}_{k+1}(e) . \tag{2.24}$$

It turns out that the map $e \mapsto \text{ch}_*(e)$ leads to a well defined map from the K theory group $K_0(\mathcal{A})$ to cyclic homology of $\mathcal{A}$ (in fact the correct receptacle is period cyclic homology [44]). In Section 6 below, we shall construct some interesting examples of this Chern character on quantum spheres. For the remaining part of this Section we shall use it to define some 'even' dimensional noncommutative algebras (including spheres).

For any pair of integers m, r we shall construct a universal algebra $\mathcal{A}_{m,r}$ as follows. We let $\mathcal{A}_{m,r}$ be generated by the r^2 elements e_{ij}, $i, j \in \{1, \ldots, r\}$, $e = [e_{ij}]$ on which we first impose the relations stating that e is an idempotent

$$e^2 = e. \tag{2.25}$$

We impose additional relations by requiring the vanishing of all 'lower degree' components of the Chern character of e,

$$\text{ch}_k(e) = 0 , \quad \forall k < m . \tag{2.26}$$

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Then, an admissible morphism from $\mathcal{A}_{m,r}$ to an arbitrary algebra $\mathcal{B}$,

$$\rho : \mathcal{A}_{m,r} \to \mathcal{B}, \tag{2.27}$$

is given by the $\rho(e_{ij}) \in \mathcal{B}$ which fulfill $\rho(e)^2 = \rho(e)$, and

$$\mathrm{ch}_k(\rho(e)) = 0 \ , \quad \forall \, k < m. \tag{2.28}$$

We define the algebra $\mathcal{A}_{m,r}$ as the quotient of the algebra defined by (2.25) by the intersection of kernels of the admissible morphisms ρ. Elements of the algebra $\mathcal{A}_{m,r}$ can be represented as polynomials in the generators e_{ij} and to prove that such a polynomial $P(e_{ij})$ is non zero in $\mathcal{A}_{m,r}$ one must construct a solution to the above equations for which $P(e_{ij}) \neq 0$.

To get a C*-algebra we endow $\mathcal{A}_{m,r}$ with the involution given by,

$$(e_{ij})^* = e_{ji} \tag{2.29}$$

which means that $e = e^*$ in $\mathrm{Mat}_r(\mathcal{A})$, i.e. e is a projection in $\mathrm{Mat}_r(\mathcal{A})$ (or equivalently, a self-adjoint idempotent). We define a norm by,

$$\|P\| = \mathrm{Sup} \ \|(\pi(P))\| \tag{2.30}$$

where π ranges through all representations of the above equations on Hilbert spaces. Such a π is given by a Hilbert space $\mathcal{H}$ and a self-adjoint idempotent,

$$E \in \mathrm{Mat}_r(\mathcal{L}(\mathcal{H})) \, , \ E^2 = E \, , \ E = E^* \tag{2.31}$$

such that (2.28) holds for $\mathcal{B} = \mathcal{L}(\mathcal{H})$. For any polynomial $P(e_{ij})$ the quantity (2.30), i.e. the supremum of the norms, $\|P(E_{ij})\|$ is finite.

We let $A_{m,r}$ be the universal C*-algebra obtained as the completion of $\mathcal{A}_{m,r}$ for the above norm.

2.3 Noncommutative algebras from unitaries

In the odd case, more than projections one rather needs unitary elements and the formulæ for the *odd* (reduced) Chern character in cyclic homology are similar to those above. The Chern character of a unitary $u \in \mathrm{Mat}_r(\mathcal{A})$ is a formal sum of chains

$$\mathrm{ch}_*(u) = \sum_k \mathrm{ch}_k(u) \ , \tag{2.32}$$

with the component $\mathrm{ch}_{n+\frac{1}{2}}(u)$ as element of $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes(2n-1)}$ given by

$$\begin{aligned}
\mathrm{ch}_{k+\frac{1}{2}}(u) \ = \ & \lambda_k \left(u_{i_1}^{i_0} \otimes (u^*)_{i_2}^{i_1} \otimes u_{i_3}^{i_2} \otimes \cdots \otimes (u^*)_{i_0}^{i_{2k+1}} \right. \\
& \left. - (u^*)_{i_1}^{i_0} \otimes u_{i_2}^{i_1} \otimes (u^*)_{i_3}^{i_2} \otimes \cdots \otimes u_{i_0}^{i_{2k+1}} \right) \ ,
\end{aligned} \tag{2.33}$$

and λ_k suitable normalization constants. Again $\mathrm{ch}_*(u)$ defines a cycle in the reduced (b, B)-bicomplex of cyclic homology [9, 10, 44, 13],

$$(b + B) \, \mathrm{ch}_*(u) = 0 \ , \quad B \, \mathrm{ch}_{k+\frac{1}{2}}(e) = b \, \mathrm{ch}_{k+\frac{1}{2}+1}(e) \, , \tag{2.34}$$

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and the map $u \mapsto \mathrm{ch}_*(u)$ leads to a well defined map from the K theory group $K_1(\mathcal{A})$ to (in fact periodic) cyclic homology.

For any pair of integers m, r we can define $\mathcal{B}_{m,r}$ to be the universal algebra generated by the r^2 elements u_{ij}, $i, j \in \{1, \ldots, r\}$, $u = [u_{ij}]$ and we impose as above the relations

$$\mathrm{ch}_{k+\frac{1}{2}}(\rho(u)) = 0 \qquad \forall \, k < m \, . \tag{2.35}$$

To get a C*-algebra we endow $\mathcal{B}_{m,r}$ with the involution given by,

$$u \, u^* = u^* \, u = 1, \tag{2.36}$$

which means that u is a unitary in $\mathrm{Mat}_r(\mathcal{A})$. As before, we define a norm by,

$$\|P\| = \mathrm{Sup} \, \|(\pi(P))\| \tag{2.37}$$

where π ranges through all representations of the above equations in Hilbert space. We let $B_{m,r}$ be the universal C*-algebra obtained as the completion of $\mathcal{B}_{m,r}$ for the above norm.

3   Fredholm modules and spectral triples

As we have mentioned in Section 2, the Chern characters $\mathrm{ch}_*(x)$ leads to well defined maps from the K theory groups $K_*(\mathcal{A})$ to (period) cyclic homology. The dual Chern characters, ch^*, of even and odd Fredholm modules provides similar maps to (period) cyclic cohomology.

3.1   Fredholm modules and index theorems

A Fredholm module can be thought of as an abstract elliptic operator. The full fledged theory started with Atiyah and culminated in the KK-theory of Kasparov and the cyclic cohomology of Connes. Here we shall only mention the few facts that we shall need later on.

Let $\mathcal{A}$ be an algebra with involution. An odd Fredholm module [9] over $\mathcal{A}$ consists of
1) a representation ψ of the algebra $\mathcal{A}$ on an Hilbert space $\mathcal{H}$;
2) an operator F on $\mathcal{H}$ such that

$$\begin{aligned} F^2 = \mathbb{I} \, , \quad F^* = F \, , \\ [F, \psi(a)] \in \mathcal{K} \, , \quad \forall \, a \in \mathcal{A} \, , \end{aligned} \tag{3.1}$$

where $\mathcal{K}$ are the compact operators on $\mathcal{H}$.
An even Fredholm module has also a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$, $\gamma^* = \gamma$, $\gamma^2 = \mathbb{I}$, such that

$$\begin{aligned} F\gamma + \gamma F = 0 \, , \\ \psi(a)\gamma - \gamma\psi(a) = 0 \, , \quad \forall \, a \in \mathcal{A} \, . \end{aligned} \tag{3.2}$$

In fact, often the first of conditions (3.1) needs to be weakened somehow to $F^2 - \mathbb{I} \in \mathcal{K}$. With an even module we shall indicate with $\mathcal{H}^\pm$ and $\psi^\pm$ the component of the Hilbert space and of the representation with respect to the grading.

Given any positive integer r, one can extend the previous modules to a Fredholm module $(\mathcal{H}_r, F_r)$ over the algebra $\mathrm{Mat}_r(\mathcal{A}) = \mathcal{A} \otimes \mathrm{Mat}_r(\mathbb{C})$ by a simple procedure

$$\mathcal{H}_r = \mathcal{H} \otimes \mathbb{C}^r \ , \quad \psi_r = \psi \otimes \mathrm{id} \ , \quad \mathbb{F}_r = F \otimes \mathbb{I}_r \ , \tag{3.3}$$

and $\gamma_r = \gamma \otimes \mathbb{I}_r$, for an even module.

The importance of Fredholm modules is testified by the following theorem which can be associated with the names of Atiyah and Kasparov [2, 37],

Theorem 1
a) *Let $(\mathcal{H}, F, \gamma)$ be an even Fredholm module over the algebra $\mathcal{A}$. And let $e \in \mathrm{Mat}_r(\mathcal{A})$ be a projection $e^2 = e = e^*$. Then we have a Fredholm operator*

$$\psi_r^-(e) F_r \psi_r^+(e) : \psi_r^+(e)\mathcal{H}_r \to \psi_r^-(e)\mathcal{H}_r \ , \tag{3.4}$$

whose index depends only on the class of the projection e in the K-theory of $\mathcal{A}$. Thus we get an additive map

$$\varphi : K_0(\mathcal{A}) \to \mathbb{Z} \ , $$
$$\varphi([e]) = \mathrm{Index}\left(\psi_r^-(e) F_r \psi_r^+(e)\right) \ . \tag{3.5}$$

b) *Let $(\mathcal{H}, F)$ be an odd Fredholm module over the algebra $\mathcal{A}$, and take the projection $E = \frac{1}{2}(\mathbb{I} + F)$. Let $u \in \mathrm{Mat}_r(\mathcal{A})$ be unitary $uu^* = u^*u = \mathbb{I}$. Then we have a Fredholm operator*

$$E_r \psi_r(u) E_r : E_r \mathcal{H}_r \to E_r \mathcal{H}_r \ , \tag{3.6}$$

whose index depends only on the class of the unitary u in the K-theory of $\mathcal{A}$. Thus we get an additive map

$$\varphi : K_1(\mathcal{A}) \to \mathbb{Z} \ , $$
$$\varphi([u]) = \mathrm{Index}\left(E_r \psi_r(u) E_r\right) \ . \tag{3.7}$$

If $\mathcal{A}$ is a C*-algebra, both in the even and the odd cases, the index map φ only depends on the K-*homology* class

$$[(\mathcal{H}, F)] \in KK(\mathcal{A}, \mathbb{C}) \ , \tag{3.8}$$

of the Fredholm module in the Kasparov KK group, $K^*(\mathcal{A}) = KK(\mathcal{A}, \mathbb{C})$, which is the abelian group of stable homotopy classes of Fredholm modules over $\mathcal{A}$ [37].
Both in the even and odd cases, the index pairings (3.5) and (3.7) can be given as [10]

$$\varphi(x) = \langle \mathrm{ch}^*(\mathcal{H}, F), \mathrm{ch}_*(x) \rangle \ , \quad x \in K_*(\mathcal{A}) \ , \tag{3.9}$$

via the Chern characters

$$\mathrm{ch}^*(\mathcal{H}, F) \in HC^*(\mathcal{A}), \quad \mathrm{ch}_*(x) \in HC_*(\mathcal{A}) \ , \tag{3.10}$$

and the pairing between cyclic cohomology $HC^*(\mathcal{A})$ and cyclic homology $HC_*(\mathcal{A})$ of the algebra $\mathcal{A}$.

The Chern character $\mathrm{ch}_*(x)$ in homology is given by (2.22) and (2.32) in the even and odd case respectively. As for the Chern character $\mathrm{ch}^*(x)$ in cohomology we shall give some fundamentals in the next Section.

3.2 The Chern characters of Fredholm modules

For the general theory we refer to [10]. In Section 6 we shall construct some interesting examples of these Chern characters on quantum spheres. Additional examples have been constructed in [35].

We recall [56] that on a Hilbert space $\mathcal{H}$ and with $\mathcal{K}$ denoting the compact operators one defines, for $p \in [1, \infty[$, the Schatten p-class, $\mathcal{L}^p$, as the ideal of compact operators for which $\operatorname{Tr} T$ is finite: $\mathcal{L}^p = \{T \in \mathcal{K} : \operatorname{Tr} T < \infty\}$. Then, the Hölder inequality states that $\mathcal{L}^{p_1} \mathcal{L}^{p_2} \cdots \mathcal{L}^{p_k} \subset \mathcal{L}^p$, with $p^{-1} = \sum_{j=1}^{k} p_j^{-1}$.

Let now $(\mathcal{H}, F)$ be Fredholm module (even or odd) over the algebra $\mathcal{A}$. We say that $(\mathcal{H}, F)$ is *p-summable* if

$$[F, \psi(a)] \in \mathcal{L}^p , \quad \forall \, a \in \mathcal{A} . \tag{3.11}$$

For simplicity, in the rest of this section, we shall drop the symbol ψ which indicates the representation on $\mathcal{A}$ on $\mathcal{H}$. The idea is then to construct 'quantized differential forms' and integrate (via a trace) forms of degree higher enough so that they belong to $\mathcal{L}^1$. In fact, one need to introduce a conditional trace. Given an operator T on $\mathcal{H}$ such that $FT + TF \in \mathcal{L}^1$, one defines

$$\operatorname{Tr}' T := \frac{1}{2} \operatorname{Tr} F(FT + TF) ; \tag{3.12}$$

note that, if $T \in \mathcal{L}^1$ then $\operatorname{Tr} T = \operatorname{Tr}' T$ by cyclicity of the trace.

Let now n be a nonnegative integer and let $(\mathcal{H}, F)$ be Fredholm module over the algebra $\mathcal{A}$. We take this module to be *even* or *odd* according to whether n is even or odd; and we shall also take it to be $(n + 1)$-summable. We shall construct a so called n-dimensional *cycle* $(\Omega^* = \oplus_k \Omega^k, d, \int)$ over the algebra $\mathcal{A}$. Elements of Ω^k are *quantized differential forms*: $\Omega^0 = \mathcal{A}$ and for $k > 0$, Ω^k is the linear span of operators of the form

$$\omega = a_0[F, a_1] \cdots [F, a_n] , \quad a_j \in \mathcal{A} . \tag{3.13}$$

By the assumption of summability, Hölder inequality gives that $\Omega^k \subset \mathcal{L}^{\frac{n+1}{k}}$. The product in Ω^* is just the product of operators $\omega\omega' \in \Omega^{k+k'}$ for any $\omega \in \Omega^k$ and $\omega' \in \Omega^{k'}$. The differential $d : \Omega^k \to \Omega^{k+1}$ is defined by

$$d\omega = F\omega - (-1)^k \omega F , \quad \omega \in \Omega^k , \tag{3.14}$$

and $F^2 = 1$ implies both $d^2 = 0$ and the fact that d is a graded derivation

$$d(\omega\omega') = (d\omega)\omega' + (-1)^k \omega d\omega' , \quad \omega \in \Omega^k , \quad \omega' \in \Omega^{k'} . \tag{3.15}$$

Finally, one defines a trace in degree n by,

$$\int : \Omega^n \to \mathbb{C} , \tag{3.16}$$

which is both closed ($\int d\omega = 0$) and graded ($\int \omega\omega' = (-1)^{kk'} \int \omega'\omega$).

Let us first consider the case n is odd. With $\omega \in \Omega^n$ one defines

$$\int \omega := \operatorname{Tr}' \omega = \frac{1}{2} \operatorname{Tr} F(F\omega + \omega F)) = \frac{1}{2} \operatorname{Tr} F d\omega , \tag{3.17}$$

which is well defined since $Fd\omega \in \mathcal{L}^1$.

If n is even and γ is the grading, with $\omega \in \Omega^n$ one defines

$$\int \omega := \mathrm{Tr}' \, \gamma\omega = \frac{1}{2} \, \mathrm{Tr} \, F(F\gamma\omega + \gamma\omega F)) = \frac{1}{2} \, \mathrm{Tr} \, \gamma F d\omega \, , \tag{3.18}$$

(remember that $F\gamma = -\gamma F$); this is again well defined since $\gamma F d\omega \in \mathcal{L}^1$. One straight-forwardly proves closeness and graded cyclicity of both the integrals (3.17) and (3.18).

The *character* of the Fredholm module is the cyclic cocycle $\tau^n \in Z^n_\lambda(\mathcal{A})$ given by,

$$\tau^n(a_0, a_1, \cdots, a_n) := \int a_0 da_1 \cdots da_n \, , \quad a_j \in \mathcal{A}; \tag{3.19}$$

explicitly,

$$\tau^n(a_0, a_1, \cdots, a_n) = \mathrm{Tr}' \, a_0[F, a_1], \cdots, [F, a_n] \, , \quad n \text{ odd} \, , \tag{3.20}$$

$$\tau^n(a_0, a_1, \cdots, a_n) = \mathrm{Tr}' \, \gamma \, a_0[F, a_1], \cdots, [F, a_n] \, , \quad n \text{ even} \, . \tag{3.21}$$

In both cases one checks closure, $b\tau^n = 0$, and cyclicity, $\lambda\tau^n = (-1)^n\tau^n$.

We see that there is ambiguity in the choice of the integer n. Given a Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$, the parity of n is fixed by for its precise value there is only a lower bound determined by the $(n+1)$-summability. Indeed, since $\mathcal{L}^{p_1} \subset \mathcal{L}^{p_2}$ if $p_1 \leq p_2$, one could replace n by $n + 2k$ with k any integer. Thus one gets a sequence of cyclic cocycle $\tau^{n+2k} \in Z^{n+2k}_\lambda(\mathcal{A}), k \geq 0$, with the same parity. The crucial fact is that the cyclic cohomology classes of these cocycles are related by the periodicity operator S in (2.20). The characters τ^{n+2k} satisfy

$$S[\tau^m]_\lambda = c_m[\tau^{m+2}]_\lambda \, , \quad \text{in} \quad HC^{m+2}(\mathcal{A}) \, , \quad m = n + 2k \, , \quad k \geq 0 \, , \tag{3.22}$$

with c_m a constant depending on m (one could get rid of these constants by suitably normalizing the characters in (3.20) and (3.21)). Therefore, the sequence $\{\tau^{n+2k}\}_{k\geq 0}$ determine a well defined class $[\tau^F]$ in the periodic cyclic cohomology $HP^0(\mathcal{A})$ or $HP^1(\mathcal{A})$ according to whether n is even or odd. The class $[\tau^F]$ is the Chern character of the Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ in periodic cyclic cohomology.

3.3 Spectral triples and index theorems

As already mentioned, a noncommutative geometry is described by a spectral triple [10]

$$(\mathcal{A}, \mathcal{H}, D) \, . \tag{3.23}$$

Here $\mathcal{A}$ is an algebra with involution, together with a representation ψ of $\mathcal{A}$ as bounded operators on a Hilbert space $\mathcal{H}$ as bounded operators, and D is a self-adjoint operator with compact resolvent and such that,

$$[D, \psi(a)] \text{ is bounded } \forall \, a \in \mathcal{A} \, . \tag{3.24}$$

An even spectral triple has also a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$, $\gamma^* = \gamma$, $\gamma^2 = \mathbb{I}$, with the additional properties,

$$D\gamma + \gamma D = 0 \, ,$$
$$\psi(a)\gamma - \gamma\psi(a) = 0 \, , \quad \forall \, a \in \mathcal{A} \, . \tag{3.25}$$

Given a spectral triple there is associated a fredholm module with the operator F just given by the sign of D, $F = D|D|^{-1}$ (if the kernel of D in not trivial one can still adjust things and define such an F).

The operator D plays in general the role of the Dirac operator [43] in ordinary Riemannian geometry. It specifies both the K-homology fundamental class (cf. [10]), as well as the metric on the state space of $\mathcal{A}$ by

$$d(\varphi, \psi) = \text{Sup} \left\{ |\varphi(a) - \psi(a)|; \|[D, a]\| \leq 1 \right\} . \tag{3.26}$$

What holds things together in this spectral point of view on noncommutative geometry is the nontriviality of the pairing between the K-theory of the algebra $\mathcal{A}$ and the K-homology class of D. There are index maps as with Fredholm modules above,

$$\varphi : K_*(\mathcal{A}) \to \mathbb{Z} \tag{3.27}$$

and the maps φ given by expressions like (3.5) and (3.7) with the operator D replacing the operator F there.

An operator theoretic index formula [10], [18], [33] expresses the above index pairing (3.27) by explicit *local* cyclic cocycles on the algebra $\mathcal{A}$. These local formulas become extremely simple in the special case where only the top component of the Chern character $\text{ch}_*(e)$ in cyclic homology fails to vanish. This is easy to understand in the analogous simpler case of ordinary manifolds since the Atiyah-Singer index formula gives the integral of the product of the Chern character $\text{ch}(E)$, of the bundle E over the manifold M, by the index class; if the only component of $\text{ch}(E)$ is ch_n, $n = \frac{1}{2} \dim M$ only the 0-dimensional component of the index class is involved in the index formula.

For instance, in the even case, provided the components $\text{ch}_k(e)$ all vanish for $k < n$ the index formula reduces to the following,

$$\varphi(e) = (-1)^n \fint \gamma \left(e - \frac{1}{2} \right) [D, e]^{2n} D^{-2n} . \tag{3.28}$$

Here, e is a projection $e^2 = e = e^*$, γ is the $\mathbb{Z}/2$ grading of $\mathcal{H}$ as above, the resolvent of D is of order $\frac{1}{2n}$ (i.e. its characteristic values μ_k are $0(k^{-\frac{1}{2n}})$) and $\fint$ is the coefficient of the logarithmic divergency in the ordinary operator trace [27] [63]. There is a similar formula for the odd case.

Example 2 *The Canonical Triple over a Manifold*
The basic example of spectral triple is the canonical triple on a closed n-dimensional Riemannian spin manifold (M, g). A spin manifold is a manifold on which it is possible to construct principal bundles having the groups $Spin(n)$ as structure groups. A manifold admits a spin structure if and only if its second Stiefel-Whitney class vanishes [43].

The canonical spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over the manifold M is as follows:
1) $\mathcal{A} = C^\infty(M)$ is the algebra of complex valued smooth functions on M.
2) $\mathcal{H} = L^2(M, \mathcal{S})$ is the Hilbert space of square integrable sections of the irreducible, rank $2^{[n/2]}$, spinor bundle over M ; its elements are spinor fields over M. The scalar product in $L^2(M, \mathcal{S})$ is the usual one of the measure $d\mu(g)$ of the metric g, $(\psi, \phi) = \int d\mu(g)\overline{\psi(x)} \cdot \phi(x)$, with the pointwise scalar product in the spinor space being the natural one in $\mathbb{C}^{2^{[n/2]}}$.
3) D is the Dirac operator of the Levi-Civita connection of the metric g. It can be written locally as

$$D = \gamma^\mu(x)(\partial_\mu + \omega_\mu^{\mathcal{S}}) , \tag{3.29}$$

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where ω_μ^S is the lift of the Levi-Civita connection to the bundle of spinors. The curved gamma matrices $\{\gamma^\mu(x)\}$ are Hermitian and satisfy

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g(dx^\mu, dx^n) = 2g^{\mu\nu} \ , \quad \mu, \nu = 1, \dots, n \ . \tag{3.30}$$

The elements of the algebra $\mathcal{A}$ act as multiplicative operators on $\mathcal{H}$,

$$(f\psi)(x) =: f(x)\psi(x) \ , \quad \forall \ f \in \mathcal{A} \ , \psi \in \mathcal{H} \ . \tag{3.31}$$

For this triple, the distance in (3.26) is the geodesic distance on the manifold M of the metric g.

An additional important ingredient is provided by a *real structure*. In the context of the canonical triple, this is given by J, the charge conjugation operator, which is an antilinear isometry of $\mathcal{H}$. We refer to [10] for all details; a friendly introduction is in [39].

4 Examples of Isospectral Deformations

We shall now construct some examples of (a priori noncommutative) spaces $\mathrm{Gr}_{m,r}$ such that

$$A_{m,r} = C(\mathrm{Gr}_{m,r}) \qquad \text{or} \qquad B_{m,r} = C(\mathrm{Gr}_{m,r}), \tag{4.1}$$

according to even or odd dimensions, with the C*-algebras $A_{m,r}$ and $B_{m,r}$ defined at the end of Section 2.2 and Section 2.3, and associated with the vanishing of the 'lower degree' components of the Chern character of an idempotent and of a unitary respectively.

4.1 Spheres in dimension 2

The simplest case is $m = 1$, $r = 2$. We have then

$$e = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \tag{4.2}$$

and the condition (2.28) just means that

$$e_{11} + e_{22} = 1 \tag{4.3}$$

while (2.25) means that

$$e_{11}^2 + e_{12}\,e_{21} = e_{11} \ , \ \ e_{11}\,e_{12} + e_{12}\,e_{22} = e_{12} \ , \tag{4.4}$$
$$e_{21}\,e_{11} + e_{22}\,e_{21} = e_{21} \ , \ \ e_{21}\,e_{12} + e_{22}^2 = e_{22} \ .$$

By (4.3) we get $e_{11} - e_{11}^2 = e_{22} - e_{22}^2$, so that (4.4) shows that $e_{12}\,e_{21} = e_{21}\,e_{12}$. We also see that e_{12} and e_{21} both commute with e_{11}. This shows that $\mathcal{A}_{1,2}$ is commutative and allows to check that $\mathrm{Gr}_{1,2} = S^2$ is the 2-sphere. Thus $\mathrm{Gr}_{1,2}$ is an ordinary commutative space.

4.2 Spheres in dimension 4

Next, we move on to the case $m = 2$, $r = 4$.

Note first that the notion of admissible morphism is a non trivial piece of structure on $\mathrm{Gr}_{2,4}$ since, for instance, the identity map is not admissible [15].

Commutative solutions were found in [13] with the commutative algebra $A = C(S^4)$ and an admissible surjection $A_{2,4} \to C(S^4)$, where the sphere S^4 appears naturally as quaternionic projective space, $S^4 = P_1(\mathbb{H})$.

In [17] we found noncommutative solutions, showing that the algebra $A_{2,4}$ is noncommutative, and we constructed explicit admissible surjections,

$$A_{2,4} \to C(S^4_\theta) \tag{4.5}$$

where S^4_θ is the noncommutative 4-sphere we are about to describe and whose form is dictated by natural deformations of the ordinary 4-sphere, similar in spirit to the standard deformation of the torus $\mathbb{T}^2$ to the noncommutative torus $\mathbb{T}^2_\theta$. In fact, as will become evident later on, noncommutative tori in arbitrary dimensions play a central role in the deformations.

We first determine the algebra generated by the usual matrices $\mathrm{Mat}_4(\mathbb{C})$ and a projection $e = e^* = e^2$ such that $\mathrm{ch}_0(e) = 0$ as above and whose matrix expression is of the form,

$$[e^{ij}] = \frac{1}{2} \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \tag{4.6}$$

where each q_{ij} is a 2×2 matrix of the form,

$$q = \begin{pmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{pmatrix} , \tag{4.7}$$

and $\lambda = \exp(2\pi i\theta)$ is a complex number of modulus one (different from -1 for convenience). Since $e = e^*$, both q_{11} and q_{22} are self-adjoint, moreover since $\mathrm{ch}_0(e) = 0$, we can find $z = z^*$ such that,

$$q_{11} = \begin{pmatrix} 1+z & 0 \\ 0 & 1+z \end{pmatrix} , \quad q_{22} = \begin{pmatrix} 1-z & 0 \\ 0 & 1-z \end{pmatrix} . \tag{4.8}$$

We let $q_{12} = \begin{pmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{pmatrix}$, we then get from $e = e^*$,

$$q_{21} = \begin{pmatrix} \alpha^* & -\overline{\lambda}\beta \\ \beta^* & \alpha \end{pmatrix} . \tag{4.9}$$

We thus see that the commutant $\mathcal{A}_\theta$ of $\mathrm{Mat}_4(\mathbb{C})$ is generated by z, α, β and we first need to find the relations imposed by the equality $e^2 = e$. In terms of the matrix

$$e = \frac{1}{2} \begin{pmatrix} 1+z & q \\ q^* & 1-z \end{pmatrix} , \tag{4.10}$$

the equation $e^2 = e$ means that $z^2 + qq^* = 1$, $z^2 + q^*q = 1$ and $[z, q] = 0$. This shows that z commutes with α, β, α^* and β^* and since $qq^* = q^*q$ is a diagonal matrix

$$\alpha\alpha^* = \alpha^*\alpha , \ \ \alpha\beta = \lambda\beta\alpha , \ \ \alpha^*\beta = \overline{\lambda}\beta\alpha^* , \ \ \beta\beta^* = \beta^*\beta \tag{4.11}$$

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so that the generated algebra $\mathcal{A}_\theta$ is not commutative for λ different from 1. The only further relation, besides $z = z^*$, is a sphere relation

$$\alpha\alpha^* + \beta\beta^* + z^2 = 1 \,. \tag{4.12}$$

We denote by S_θ^4 the corresponding noncommutative space defined by 'duality', so that its algebra of polynomial functions is $\mathcal{A}(S_\theta^4) = \mathcal{A}_\theta$. This algebra is a deformation of the commutative $*$-algebra $\mathcal{A}(S^4)$ of complex polynomial functions on the usual sphere S^4 to which it reduces for $\theta = 0$.

The projection e given in (4.10) is clearly an element in the matrix algebra $\mathrm{Mat}_4(\mathcal{A}_\theta) \simeq \mathrm{Mat}_4(\mathbb{C}) \otimes \mathcal{A}_\theta$. Then, it naturally acts on the free $\mathcal{A}_\theta$-module $\mathcal{A}_\theta^4 \simeq \mathbb{C}^4 \otimes \mathcal{A}_\theta$ and one gets as its range a finite projective module which can be thought of as the module of 'section of a vector bundle' over S_θ^4. The module $e\mathcal{A}_\theta^4$ is a deformation of the usual [3] complex rank 2 instanton bundle over S^4 to which it reduces for $\theta = 0$ [40].

For the sphere S_θ^4 the deformed instanton has correct characteristic classes. The fact that $\mathrm{ch}_0(e)$ has been imposed from the very beginning and could be interpreted as stating the fact that the projection and the corresponding module (the 'vector bundle') has complex rank equal to 2. Next, we shall check that the two dimensional component $\mathrm{ch}_1(e)$ of the Chern character, automatically vanishes as an element of the (reduced) (b, B)-bicomplex.

With $q = \left(\begin{smallmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{smallmatrix} \right)$, we get,

$$\mathrm{ch}_1(e) = \frac{1}{2^3} \Big\langle z \left(dq\, dq^* - dq^*\, dq \right) + q \left(dq^*\, dz - dz\, dq^* \right) + q^* \left(dz\, dq - dq\, dz \right) \Big\rangle$$

where the expectation in the right hand side is relative to $\mathrm{Mat}_2(\mathbb{C})$ (it is a partial trace) and we use the notation d instead of the tensor notation. The diagonal elements of $\omega = dq\, dq^*$ are

$$\omega_{11} = d\alpha\, d\alpha^* + d\beta\, d\beta^* \,, \quad \omega_{22} = d\beta^*\, d\beta + d\alpha^*\, d\alpha$$

while for $\omega' = dq^*\, dq$ we get,

$$\omega'_{11} = d\alpha^*\, d\alpha + d\beta\, d\beta^* \,, \quad \omega'_{22} = d\beta^*\, d\beta + d\alpha\, d\alpha^* \,.$$

It follows that, since z is diagonal,

$$\Big\langle z \left(dq\, dq^* - dq^*\, dq \right) \Big\rangle = 0 \,. \tag{4.13}$$

The diagonal elements of $q\, dq^*\, dz = \rho$ are

$$\rho_{11} = \alpha\, d\alpha^*\, dz + \beta\, d\beta^*\, dz \,, \quad \rho_{22} = \beta^*\, d\beta\, dz + \alpha^*\, d\alpha\, dz$$

while for $\rho' = q^*\, dq\, dz$ they are

$$\rho'_{11} = \alpha^*\, d\alpha\, dz + \beta\, d\beta^*\, dz \,, \quad \rho'_{22} = \beta^*\, d\beta\, dz + \alpha\, d\alpha^*\, dz \,.$$

Similarly for $\sigma = q\, dz\, dq^*$ and $\sigma' = q^*\, dz\, dq$ one gets the required cancellations so that,

$$\mathrm{ch}_1(e) = 0 \,, \tag{4.14}$$

Summing up we thus get that the element $e \in C^\infty(S_\theta^4, \mathrm{Mat}_4(\mathbb{C}))$ given in (4.10) is a self-adjoint idempotent, $e = e^2 = e^*$, and satisfies $\mathrm{ch}_k(e) = 0 \; \forall \, k < 2$. Moreover, $\mathrm{Gr}_{2,4}$ is a noncommutative space and $S_\theta^4 \subset \mathrm{Gr}_{2,4}$.

Since $\mathrm{ch}_1(e) = 0$, it follows that $\mathrm{ch}_2(e)$ is a Hochschild cycle which will play the role of the round volume form on S_θ^4 and that we shall now compute. With the above notations one has,

$$\mathrm{ch}_2(e) = \frac{1}{2^5} \left\langle \begin{pmatrix} z & q \\ q^* & -z \end{pmatrix} \begin{pmatrix} dz & dq \\ dq^* & -dz \end{pmatrix}^4 \right\rangle . \tag{4.15}$$

The direct computation gives the Hochschild cycle $\mathrm{ch}_2(e)$ as a sum of five components

$$\mathrm{ch}_2(e) = z \, c_z + \alpha \, c_\alpha + \alpha^* \, c_{\alpha^*} + \beta \, c_\beta + \beta^* \, c_{\beta^*} ; \tag{4.16}$$

where the components $c_z \, , c_\alpha \, , c_{\alpha^*} \, , c_\beta \, , c_{\beta^*}$, which are elements in the four-fold tensor product $\mathcal{A}_\theta \otimes \overline{\mathcal{A}_\theta} \otimes \overline{\mathcal{A}_\theta} \otimes \mathcal{A}_\theta$, are explicitly given in [17]. The vanishing of $b \, \mathrm{ch}_2(e)$, which has six hundred terms, can be checked directly from the commutation relations (4.11). The cycle $\mathrm{ch}_2(e)$ is totally 'λ-antisymmetric'.

Our sphere S_θ^4 is by construction the suspension of the noncommutative 3-sphere S_θ^3 whose coordinate algebra is generated by α and β as above and say the special value $z = 0$. This 3-sphere is part of a family of spheres that we shall describe in the next Section.

Had we taken the deformation parameter to be real, $\lambda = q \in \mathbb{R}$, the corresponding 3-sphere S_q^3 would coincide with the quantum group $SU(2)_q$. Similarly, had we taken the deformation parameter in S_θ^4 to be real like in [24] we would have obtained a different deformation S_q^4 of the commutative sphere S^4, whose algebra is different from the above one. More important, the component $\mathrm{ch}_1(e)$ of the Chern character would not vanish [23].

4.3 Spheres in dimension 3

Odd dimensional spaces, in particular spheres, are constructed out of unitaries rather than projections [17, 15, 16].

Let us consider the lowest dimensional case for which $m = 2, r = 2$. We shall use the convention that repeated indices are summed on. Greek indices like $\mu, \nu, \cdots$, are taken to be valued in $\{0, 1, 2, 3\}$ while latin indices like $j, k, \cdots$, are taken to be valued in $\{1, 2, 3\}$.

We are then looking for an algebra $\mathcal{B}$ such that

1) $\mathcal{B}$ is generated as a unital $*$-algebra by the entries of a unitary matrix

$$u \in \mathrm{Mat}_2(\mathcal{B}) \simeq \mathrm{Mat}_2(\mathbb{C}) \otimes \mathcal{B}, \quad uu^* = u^*u = 1, \tag{4.17}$$

2) the unitary u satisfies the additional condition

$$\mathrm{ch}_{\frac{1}{2}}(u) := \sum u_i^j \otimes (u^*)_i^j - (u^*)_i^j \otimes u_i^j = 0 . \tag{4.18}$$

Let us take as 'generators' of $\mathcal{B}$ elements $z^\mu, z^{\mu*}$, $\mu \in \{0, 1, 2, 3\}$. Then using ordinary Pauli matrices σ_k, $k \in \{1, 2, 3\}$, an element in $u \in \mathrm{Mat}_2(\mathcal{B})$ can be written as

$$u = \mathbb{I}_2 z^0 + \sigma_k z^k . \tag{4.19}$$

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The requirement that u be unitaries give the following conditions on the generators

$$z^k z^{0*} - z^0 z^{k*} + \varepsilon_{klm} z^l z^{m*} = 0 \ ,$$
$$z^{0*} z^k - z^{k*} z^0 + \varepsilon_{klm} z^{l*} z^m = 0 \ ,$$
$$\sum_{\mu=0}^{3} (z^\mu z^{\mu*} - z^{\mu*} z^\mu) = 0 \ , \tag{4.20}$$

together with the condition that

$$\sum_{\mu=0}^{3} z^{\mu*} z^\nu = 1 \ . \tag{4.21}$$

Notice that the 'sphere' relation (4.21) is consistent with the relations (4.20) since the latter imply that $\sum_{\mu=0}^{3} z^{\mu*} z^\nu$ is in the center of $\mathcal{B}$.

Then, one imposes condition (4.18) which reads

$$\sum_{\mu=0}^{3} (z^{\mu*} \otimes z^\mu - z^\mu \otimes z^{\mu*}) = 0 \ , \tag{4.22}$$

and which is satisfied [15, 16] if and only if there exists a symmetric unitary matrix $\Lambda \in \mathrm{Mat}_4(\mathbb{C})$ such that

$$z^{\mu*} = \Lambda^\mu_\nu z^\nu \ . \tag{4.23}$$

Now, there is some freedom in the definition of the algebra $\mathcal{B}$ which is stated by the fact that the defining conditions 1.) and 2.) above do not change if we transform

$$z^\mu \mapsto \rho S^\mu_\nu z^\nu \ , \tag{4.24}$$

with $\rho \in \mathrm{U}(1)$ and $S \in \mathrm{SO}(4)$. Under this transformation, the matrix Λ in (4.23) transforms as

$$\Lambda \mapsto \rho^2 S^t \Lambda S \ . \tag{4.25}$$

One can then diagonalize the symmetric unitary Λ by a real rotation S and fix its first eigenvalue to be 1 by an appropriate choice of $\rho \in \mathrm{U}(1)$. So, we can take

$$\Lambda = \mathrm{diag}(1, e^{-i\varphi_1}, e^{-i\varphi_2}, e^{-i\varphi_3}) \ , \tag{4.26}$$

that is, we can put

$$z^0 = x^0, \quad z^k = e^{i\varphi_k} x^k, \quad k \in \{1, 2, 3\} \ , \tag{4.27}$$

with $e^{-i\varphi_k} \in \mathrm{U}(1)$ and $(x^\mu)^* = x^\mu$. Conditions (4.20) translate to

$$[x^0, x^k]_- \cos\varphi_k = i\ [x^l, x^m]_+ \sin(\varphi_l - \varphi_m) \ ,$$
$$[x^0, x^k]_+ \sin\varphi_k = i\ [x^l, x^m]_- \cos(\varphi_l - \varphi_m) \ , \tag{4.28}$$

with (k, l, m) the cyclic permutation of $(1, 2, 3)$ starting with $k = 1, 2, 3$ and $[x, y]_\pm = xy - yx$. There is also the sphere relation (4.21),

$$\sum_{\mu=0}^{3} (x^{\mu*})^2 = 1 \ . \tag{4.29}$$

We have therefore a three parameters family of algebras $\mathcal{B}_\varphi$ which are labelled by an element $\varphi = (e^{-i\varphi_1}, e^{-i\varphi_2}, e^{-i\varphi_3}) \in \mathbb{T}^3$. The algebras $\mathcal{B}_\varphi$ are deformations of the algebra $A(S^3)$ of polynomial functions on an ordinary 3-sphere S^3 which is obtained for the special value $\varphi = (1,1,1)$. We denote by S^3_φ the corresponding noncommutative space, so that $A(S^3_\varphi) = \mathcal{B}_\varphi$. Next, one computes $\mathrm{ch}_{\frac{3}{2}}(u_\varphi))$ and shows that is a non trivial cycle $(b\,\mathrm{ch}_{\frac{3}{2}}(u_\varphi)) = 0)$ on $\mathcal{B}_\varphi$ [15].

A special value of the parameter φ gives the 3-sphere S^3_θ described at the end of previous Section. Indeed, put $\varphi_1 = \varphi_2 = -\pi\theta$ and $\varphi_3 = 0$ and define

$$
\begin{aligned}
\alpha = x^0 + i\,x^3 \,, \quad &\alpha^* = x^0 - i\,x^3 \,, \\
\beta = x^1 + i\,x^2 \,, \quad &\beta^* = x^1 - i\,x^2 \,.
\end{aligned}
\tag{4.30}
$$

then $\alpha, \alpha^*, \beta, \beta^*$ satisfies conditions (4.11), with $\lambda = \exp(2\pi\,i\,\theta)$, together with the relation $\alpha\alpha^* + \beta\beta^* = 1$, thus defining the sphere S^3_θ of Section 4.2.

In Section 4.6 we shall describe some higher dimensional examples.

4.4   The noncommutative geometry of S^4_θ

Next we will analyze the metric structure, via a Dirac operator D, on our noncommutative 4-spheres S^4_θ. The operator D will give a solution to the following quartic equation,

$$
\left\langle \left(e - \frac{1}{2} \right) [D, e]^4 \right\rangle = \gamma
\tag{4.31}
$$

where $\langle\ \rangle$ is the projection on the commutant of 4×4 $\mathbb{C}$-matrices (in fact, it is a partial trace on the matrix entries) and $\gamma = \gamma_5$, in the present four dimensional case, is the grading operator.

Let $C^\infty(S^4_\theta)$ be the algebra of smooth functions on the noncommutative sphere S^4_θ. We shall construct a spectral triple $(C^\infty(S^4_\theta), \mathcal{H}, D)$ which describes the geometry on S^4_θ corresponding to the round metric.

In order to do that we first need to find good coordinates on S^4_θ in terms of which the operator D will be easily expressed. We choose to parametrize α, β and z as follows,

$$
\alpha = u \cos\varphi \cos\psi \,, \quad \beta = v \sin\varphi \cos\psi \,, \quad z = \sin\psi \,.
\tag{4.32}
$$

Here φ and ψ are ordinary angles with domain $0 \leq \varphi \leq \frac{\pi}{2}$, $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$, while u and v are the usual unitary generators of the algebra $C^\infty(\mathbb{T}^2_\theta)$ of smooth functions on the noncommutative 2-torus. Thus the presentation of their relations is

$$
uv = \lambda vu \,, \quad uu^* = u^*u = 1 \,, \quad vv^* = v^*v = 1 \,.
\tag{4.33}
$$

One checks that α, β, z given by (2) satisfy the basic presentation of the generators of $C^\infty(S^4_\theta)$ which thus appears as a *subalgebra* of the algebra generated (and then closed under smooth calculus) by $e^{i\varphi}$, $e^{i\psi}$, u and v.

For $\theta = 0$ the round metric is given as,

$$
G = d\alpha\,d\overline{\alpha} + d\beta\,d\overline{\beta} + dz^2
\tag{4.34}
$$

and in terms of the coordinates, φ, ψ, u, v one gets,

$$
G = \cos^2\varphi \cos^2\psi\, du\,d\overline{u} + \sin^2\varphi \cos^2\psi\, dv\,d\overline{v} + \cos^2\psi\, d\varphi^2 + d\psi^2 \,.
\tag{4.35}
$$

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Its volume form is given by

$$\omega = \frac{1}{2} \sin\varphi \cos\varphi \, (\cos\psi)^3 \, \overline{u} \, du \wedge \overline{v} \, dv \wedge d\varphi \wedge d\psi \,. \tag{4.36}$$

In terms of these rectangular coordinates we get the following simple expression for the Dirac operator,

$$\begin{aligned}
D &= (\cos\varphi \cos\psi)^{-1} \, u \, \frac{\partial}{\partial u} \, \gamma_1 + (\sin\varphi \cos\psi)^{-1} \, v \, \frac{\partial}{\partial v} \, \gamma_2 + \tag{4.37} \\
&+ \frac{\mathrm{i}}{\cos\psi} \left(\frac{\partial}{\partial \varphi} + \frac{1}{2} \cotg\varphi - \frac{1}{2} \tg\varphi \right) \gamma_3 + \mathrm{i} \left(\frac{\partial}{\partial \psi} - \frac{3}{2} \tg\psi \right) \gamma_4 \,.
\end{aligned}$$

Here γ_μ are the usual Dirac 4×4 matrices with

$$\{\gamma_\mu, \gamma_\nu\} = 2 \, \delta_{\mu\nu} \,, \quad \gamma_\mu^* = \gamma_\mu \,. \tag{4.38}$$

It is now easy to move on to the noncommutative case, the only tricky point is that there are nontrivial boundary conditions for the operator D, which are in particular antiperiodic in the arguments of both u and v. We shall just leave them unchanged in the noncommutative case, the only thing which changes is the algebra and the way it acts in the Hilbert space as we shall explain in more detail in the next section. The formula for the operator D is now,

$$\begin{aligned}
D &= (\cos\varphi \cos\psi)^{-1} \, \delta_1 \, \gamma_1 + (\sin\varphi \cos\psi)^{-1} \, \delta_2 \, \gamma_2 + \tag{4.39} \\
&+ \frac{\mathrm{i}}{\cos\psi} \left(\frac{\partial}{\partial \varphi} + \frac{1}{2} \cotg\varphi - \frac{1}{2} \tg\varphi \right) \gamma_3 + \mathrm{i} \left(\frac{\partial}{\partial \psi} - \frac{3}{2} \tg\psi \right) \gamma_4 \,.
\end{aligned}$$

where the γ_μ are the usual Dirac matrices and where δ_1 and δ_2 are the derivations of the noncommutative torus so that

$$\begin{aligned}
\delta_1(u) &= u \,, \quad \delta_1(v) = 0 \,, \tag{4.40} \\
\delta_2(u) &= 0 \,, \quad \delta_2(v) = v \,;
\end{aligned}$$

One can then check that the corresponding metric is the round one.

In order to compute the operator $\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle$ (in the tensor product by $\mathrm{Mat}_4(\mathbb{C})$) we need the commutators of D with the generators of $C^\infty(S_\theta^4)$. They are given by the following simple expressions,

$$\begin{aligned}
[D, \alpha] &= u \left\{ \gamma_1 - \mathrm{i} \sin\phi \, \gamma_3 - \mathrm{i} \cos\phi \sin\psi \, \gamma_4 \right\} \,, \tag{4.41} \\
[D, \alpha^*] &= -u^* \left\{ \gamma_1 + \mathrm{i} \sin\phi \, \gamma_3 + \mathrm{i} \cos\phi \sin\psi \, \gamma_4 \right\} \,, \\
[D, \beta] &= v \left\{ \gamma_2 + \mathrm{i} \cos\phi \, \gamma_3 - \mathrm{i} \sin\phi \sin\psi \, \gamma_4 \right\} \,, \\
[D, \beta^*] &= -v^* \left\{ \gamma_2 - \mathrm{i} \cos\phi \, \gamma_3 + \mathrm{i} \sin\phi \sin\psi \, \gamma_4 \right\} \,, \\
[D, z] &= \mathrm{i} \cos\psi \, \gamma_4 \,.
\end{aligned}$$

We check in particular that they are all bounded operators and hence that for any $f \in C^\infty(S_\theta^4)$ the commutator $[D, f]$ is bounded. Then, a long but straightforward calculation shows that equation (4.31) is valid: the operator $\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle$ is a multiple of $\gamma = \gamma_5 := \gamma_1 \gamma_2 \gamma_3 \gamma_4$. One first checks that it is equal to $\pi(\mathrm{ch}_2(e))$ where $\mathrm{ch}_2(e)$ is the Hochschild cycle in (4.16) and π is the canonical map from the Hochschild chains to operators given by

$$\pi(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 [D, a_1] \ldots [D, a_n] \,. \tag{4.42}$$

4.5   Isospectral noncommutative geometries

We shall describe fully a noncommutative geometry for S^4_θ with the couple $(\mathcal{H}, D)$ just the 'commutative' ones associated with the commutative sphere S^4; hence realizing an isospectral deformation. We shall in fact describe a very general construction of isospectral deformations of noncommutative geometries which implies in particular that any compact spin Riemannian manifold M whose isometry group has rank ≥ 2 admits a natural one-parameter isospectral deformation to noncommutative geometries M_θ. The deformation of the algebra will be performed along the lines of [52] (see also [60] and [57]).

Let us start with the canonical spectral triple $(\mathcal{A} = C^\infty(S^4), \mathcal{H}, D)$ associated with the sphere S^4. We recall that $\mathcal{H} = L^2(S^4, \mathcal{S})$ is the Hilbert space of spinors and D is the Dirac operator. Also, there is a real structure provided by J, the charge conjugation operator, which is an antilinear isometry of $\mathcal{H}$.

Recall that on the sphere S^4 there is an isometric action of the 2-torus, $\mathbb{T}^2 \subset \mathrm{Isom}(S^4)$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the usual torus. We let $U(s), s \in \mathbb{T}^2$, be the corresponding (projective) unitary representation in $\mathcal{H} = L^2(S^4, \mathcal{S})$ so that by construction

$$U(s)\, D = D\, U(s), \quad U(s)\, J = J\, U(s). \tag{4.43}$$

Also,

$$U(s)\, a\, U(s)^{-1} = \alpha_s(a), \quad \forall\, a \in \mathcal{A}, \tag{4.44}$$

where $\alpha_s \in \mathrm{Aut}(\mathcal{A})$ is the action by isometries on functions on S^4.
We let $p = (p_1, p_2)$ be the generator of the two-parameter group $U(s)$ so that

$$U(s) = \exp(2\pi i(s_1 p_1 + s_2 p_2)). \tag{4.45}$$

The operators p_1 and p_2 commute with D but anticommute with J (due to the antilinearity of the latter). Both p_1 and p_2 have half-integral spectrum,

$$\mathrm{Spec}(2\, p_j) \subset \mathbb{Z}, \quad j = 1, 2. \tag{4.46}$$

Next, we define a bigrading of the algebra of bounded operators in $\mathcal{H}$ with the operator T declared to be of bidegree (n_1, n_2) when,

$$\alpha_s(T) := U(s)\, T\, U(s)^{-1} = \exp(2\pi i(s_1 n_1 + s_2 n_2))\, T, \quad \forall\, s \in \mathbb{T}^2. \tag{4.47}$$

Any operator T of class C^∞ relative to α_s (i. e. such that the map $s \to \alpha_s(T)$ is of class C^∞ for the norm topology) can be uniquely written as a doubly infinite norm convergent sum of homogeneous elements,

$$T = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}, \tag{4.48}$$

with $\widehat{T}_{n_1, n_2}$ of bidegree (n_1, n_2) and where the sequence of norms $||\widehat{T}_{n_1, n_2}||$ is of rapid decay in (n_1, n_2).

Let now $\lambda = \exp(2\pi\, i\, \theta)$. For any operator T in $\mathcal{H}$ of class C^∞ relative to the action of $\mathbb{T}^2$ we define its left twist $l(T)$ by

$$l(T) = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}\, \lambda^{n_2 p_1}, \tag{4.49}$$

and its right twist $r(T)$ by

$$r(T) = \sum_{n_1, n_2} \lambda^{n_1 p_2} \, \widehat{T}_{n_1, n_2} \,, \tag{4.50}$$

Since $|\lambda| = 1$ and p_1, p_2 are self-adjoint, both series converge in norm. The construction involves in the case of half-integral spin the choice of a square root of λ.
One has the following,

Lemma 1
a) *Let x be a homogeneous operator of bidegree (n_1, n_2) and y be a homogeneous operator of bidegree (n_1', n_2'). Define*

$$x * y = \lambda^{n_1' n_2} \, xy \,; \tag{4.51}$$

then $l(x)l(y) = l(x * y)$.
b) *Let x and y be homogeneous operators as before. Then,*

$$l(x)\, r(y) - r(y)\, l(x) = (x\, y - y\, x)\, \lambda^{n_1'(n_2 + n_2')} \lambda^{n_2 p_1 + n_1' p_2} \,. \tag{4.52}$$

In particular, $[l(x), r(y)] = 0$ if $[x, y] = 0$.

To check a) and b) one simply uses the following commutation rule which follows from (4.47) and it is fulfilled for any homogeneous operator T of bidegree (m, n),

$$\lambda^{a p_1 + b p_2} \, T = \lambda^{am + bn} \, T \, \lambda^{a p_1 + b p_2} \,, \quad \forall a, b \in \mathbb{Z} \,. \tag{4.53}$$

The product $*$ defined in equation (4.51) extends by linearity to an associative $*$-product on the linear space of smooth operators.
One could also define a deformed 'right product'. If x is homogeneous of bidegree (n_1, n_2) and y is homogeneous of bidegree (n_1', n_2') the product is defined by

$$x *_r y = \lambda^{-n_1' n_2} \, xy \,. \tag{4.54}$$

Then, as with the previous lemma one shows that $r(x)r(y) = r(x *_r y)$.

By Lemma 1 a) one has that $l(C^\infty(S^4))$ is still an algebra and we shall identify it with (the image on the Hilbert space $\mathcal{H}$ of) the algebra $C^\infty(S_\theta^4)$ of smooth functions on the deformed sphere S_θ^4.
We can then define a new spectral triple $(l(C^\infty(S^4)) \simeq C^\infty(S_\theta^4), \mathcal{H}, D)$ where both the Hilbert space $\mathcal{H}$ and the operator D are unchanged while the algebra $C^\infty(S^4)$ is modified to $l(C^\infty(S^4)) \simeq C^\infty(S_\theta^4)$. Since D is of bidegree $(0,0)$ one has that

$$[D, l(a)] = l([D, \, a]) \tag{4.55}$$

which is enough to check that $[D, x]$ is bounded for any $x \in l(\mathcal{A})$.

Next, we also deform the real structure by twisting the charge conjugation isometry J by

$$\widetilde{J} = J \lambda^{-p_1 p_2} \,. \tag{4.56}$$

Due to the antilinearity of J one has that $\widetilde{J} = \lambda^{p_1 p_2} J$ and hence

$$\widetilde{J}^2 = J^2 \,. \tag{4.57}$$

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Lemma 2
For x homogeneous of bidegree (n_1, n_2) one has that

$$\widetilde{J} \, l(x) \, \widetilde{J}^{-1} = r(J \, x \, J^{-1}) \, . \tag{4.58}$$

For the proof one needs to check that $\widetilde{J} \, l(x) = r(J \, x \, J^{-1}) \, \widetilde{J}$. One has

$$\lambda^{-p_1 p_2} \, x = x \, \lambda^{-(p_1+n_1)(p_2+n_2)} = x \, \lambda^{-n_1 n_2} \, \lambda^{-(p_1 n_2 + n_1 p_2)} \, \lambda^{-p_1 p_2} \, . \tag{4.59}$$

Then

$$\widetilde{J} \, l(x) = J \, \lambda^{-p_1 p_2} \, x \, \lambda^{n_2 p_1} = J \, x \, \lambda^{-n_1 n_2} \, \lambda^{-n_1 p_2} \, \lambda^{-p_1 p_2} \, , \tag{4.60}$$

while

$$r(J \, x \, J^{-1}) \, \widetilde{J} = \lambda^{-n_1 p_2} \, J \, x \, J^{-1} \, J \, \lambda^{-p_1 p_2} = J \, x \, \lambda^{-n_1(p_2+n_2)} \, \lambda^{-p_1 p_2} \, . \tag{4.61}$$

Thus one gets the required equality of Lemma 2.

For $x, y \in l(\mathcal{A})$ one checks that

$$[x, \, y^0] = 0 \, , \quad y^0 = \widetilde{J} \, y^* \, \widetilde{J}^{-1} \, . \tag{4.62}$$

Indeed, one can assume that x and y are homogeneous and use Lemma 2 together with Lemma 1 a). Combining equation (4.62) with equation (4.55) one then checks the order one condition

$$[[D, \, x] \, , y^0] = 0 \, , \quad \forall \, x, y \in l(\mathcal{A}) \, . \tag{4.63}$$

Summing up, we have the following

Theorem 2
a) The spectral triple $(C^\infty(S_\theta^4), \mathcal{H}, D)$ fulfills all axioms of noncommutative manifolds.
b) Let $e \in C^\infty(S_\theta^4, \mathrm{Mat}_4(\mathbb{C}))$ be the canonical idempotent given in (4.10). The Dirac operator D fulfills

$$\left\langle \left(e - \frac{1}{2} \right) [D, e]^4 \right\rangle = \gamma$$

where $\langle \; \rangle$ is the projection on the commutant of $\mathrm{Mat}_4(\mathbb{C})$ (i.e. a partial trace) and γ is the grading operator.

Moreover, the real structure is given by the twisted involution $\widetilde{J}$ defined in (4.56). One checks using the results of [53] and [12] that Poincaré duality continues to hold for the deformed spectral triple.
Theorem 2 can be extended to all metrics on the sphere S^4 which are invariant under rotation of u and v and have the same volume form as the round metric. In fact, by paralleling the construction for the sphere described above, one can extend it quite generally [17]:

Theorem 3
Let M be a compact spin Riemannian manifold whose isometry group has rank ≥ 2 (so that one has an inclusion $\mathbb{T}^2 \subset \mathrm{Isom}(M)$). Then M admits a natural one-parameter isospectral deformation to noncommutative (spin) geometries M_θ.

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Let $(\mathcal{A}, \mathcal{H}, D)$ be the canonical spectral triple associated with a compact Riemannian spin manifold M as described in Ex. 2. Here $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M; $\mathcal{H} = L^2(M, \mathcal{S})$ is the Hilbert space of spinors and D is the Dirac operator. Finally, there is the charge conjugation operator J, an antilinear isometry of $\mathcal{H}$ which gives the real structure.

The deformed spectral triple is given by $(l(\mathcal{A}), \mathcal{H}, D)$ with $\mathcal{H} = L^2(M, \mathcal{S})$ the Hilbert space of spinors, D the Dirac operator and $l(\mathcal{A})$ is really the algebra of smooth functions on M with product deformed to a $*$-product defined in a way exactly similar to (4.51). The real structure is given by the twisted involution $\widetilde{J}$ defined as in (4.56). And again, by the results of [53] and [12], Poincaré duality continues to hold for the deformed spectral triple.

4.6 Noncommutative spherical manifolds

As we have seen, on the described deformations one changes the algebra and the way it acts on the Hilbert space while keeping the latter and the Dirac operator unchanged, thus getting isospectral deformations. From the decomposition (4.47) and the deformed product (4.51) one sees that a central role is played by tori and their noncommutative generalizations. We are now going to describe in more details this use of the noncommutative tori.

Let $\theta = (\theta_{jk} = -\theta_{kj})$ be a real antisymmetric $n \times n$ matrix. The noncommutative torus $\mathbb{T}_\theta^n$ of 'dimension' n and twist θ is the 'quantum space' whose algebra of polynomial functions $A(\mathbb{T}_\theta^n)$ is generated by n independent unitaries $u_1, \ldots, u_n$, subject to the commutation relations [8, 51]

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j. \tag{4.64}$$

The corresponding C*-algebra of continuous functions is the universal C*-algebra $C(\mathbb{T}_\theta^n)$ with the same generators and relations. There is an action τ of $\mathbb{T}^n$ on this C*-algebra. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{T}^n$, this action is given by

$$\tau(e^{2\pi i \alpha}) : u_j \mapsto e^{2\pi i \alpha_j} u_j.$$

The smooth subalgebra $C^\infty(\mathbb{T}_\theta^n)$ of $C(\mathbb{T}_\theta^n)$ under this action consists of rapidly convergent Fourier series of the form $\sum_{r \in \mathbb{Z}^n} a_r u^r$, with $a_r \in \mathbb{C}$, where

$$u^r := e^{-\pi i r_j \theta_{jk} r_k} u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}.$$

The unitary elements $\{u^r : r \in \mathbb{Z}^n\}$ form a Weyl system [33], since

$$u^r u^s = e^{\pi i r_j \theta_{jk} s_k} u^{r+s}.$$

The phase factors

$$\rho_\theta(r, s) := \exp\{\pi i r_j \theta_{jk} s_k\} \tag{4.65}$$

form a 2-cocycle for the group $\mathbb{Z}^n$, which is skew (i.e., $\rho_\theta(r, r) = 1$) since θ is skew-symmetric. This also means that $C(\mathbb{T}_\theta^n)$ may be defined as the twisted group C*-algebra $C(\mathbb{Z}^n, \rho_\theta)$.

Let now M be a compact manifold (with no boundary) carrying a smooth action σ of a torus $\mathbb{T}^n$ of dimension $n \geq 2$. By averaging the translates of a given Riemannian metric

on M over this torus, we may assume that M has a $\mathbb{T}^n$-invariant metric g, so that $\mathbb{T}^n$ acts by isometries.

The general θ-deformation of M can be accomplished in two equivalent ways. The general θ-deformation of M can be accomplished in two equivalent ways. Firstly, in [17] the deformation is given by a star product of ordinary functions along the lines of [52] (see also [60]). Indeed, the algebra $C^\infty(M)$ may be decomposed into spectral subspaces which are indexed by the dual group $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$. Now, each $r \in \mathbb{Z}^n$ labels a character $e^{2\pi i\alpha} \mapsto e^{2\pi i r \cdot \alpha}$ of $\mathbb{T}^n$, with the scalar product $r \cdot \alpha := r_1\alpha_1 + \cdots + r_n\alpha_n$. The r-th spectral subspace for the action σ of $\mathbb{T}^n$ on $C^\infty(M)$ consists of those smooth functions f_r for which

$$\sigma(e^{2\pi i\alpha})f_r = e^{2\pi i r \cdot \alpha} f_r,$$

and each $f \in C^\infty(M)$ is the sum of a unique (rapidly convergent) series $f = \sum_{r \in \mathbb{Z}^n} f_r$.

The θ-deformation of $C^\infty(M)$ may be defined by replacing the ordinary product by a *Moyal product*, defined on spectral subspaces by

$$f_r \star_\theta g_s := \rho_\theta(r, s) f_r g_s, \tag{4.66}$$

with $\rho_\theta(r, s)$ the phase factor in (4.65). Thus the deformed product is also taken to respects the $\mathbb{Z}^n$-grading of functions.

In particular, when $M = \mathbb{T}^n$ with the obvious translation action, the algebras $(C^\infty(\mathbb{T}^n), \star_\theta)$ and $C^\infty(\mathbb{T}^n_\theta)$ are isomorphic.

In the general case, we write $C^\infty(M_\theta) := (C^\infty(M), \star_\theta)$. Thus, at the level of smooth algebras the deformation is given explicitly by the star product of ordinary smooth functions. It is shown in [52] that there is a natural completion of the algebra $C^\infty(M_\theta)$ to a C*-algebra $C(M_\theta)$ whose smooth subalgebra (under the extended action of $\mathbb{T}^n$) is precisely the algebra $C^\infty(M_\theta)$.

An equivalent approach [15], is to define $C(M_\theta)$ as the fixed-point C*-subalgebra of $C(M) \otimes C(\mathbb{T}^n_\theta)$ under the action $\sigma \times \tau^{-1}$ of $\mathbb{T}^n$ defined by

$$e^{2\pi i\alpha} \cdot (f \otimes a) := \sigma(e^{2\pi i\alpha}) f \otimes \tau(e^{-2\pi i\alpha}) a;$$

that is,

$$C(M_\theta) := \left(C(M) \otimes C(\mathbb{T}^n_\theta)\right)^{\sigma \times \tau^{-1}}.$$

The smooth subalgebra is then given by

$$C^\infty(M_\theta) := \left(C^\infty(M) \widehat{\otimes} C^\infty(\mathbb{T}^n_\theta)\right)^{\sigma \times \tau^{-1}}, \tag{4.67}$$

with $\widehat{\otimes}$ denoting the appropriate (projective) tensor product of Fréchet algebras. This approach has the advantage that $C^\infty(M_\theta)$ may be determined by generators and relations with the algebra structure specified by the basic commutation relations (4.64) [15].

4.7 The θ-deformed planes and spheres in any dimensions

We shall briefly describe these classes of spaces while referring to [15] for more details.

Let $\theta = (\theta_{jk} = -\theta_{kj})$ be a real antisymmetric $n \times n$ matrix. And denote $\lambda^{jk} = e^{2\pi i \theta_{jk}}$; then we have that $\lambda^{kj} = (\lambda^{jk})^{-1}$ and $\lambda^{jj} = 1$.

Let $\mathcal{A}(\mathbb{R}_\theta^{2n})$ be the complex unital $*$-algebra generated by $2n$ elements $(z^j, z^{k*},\ j, k = 1, \ldots, n)$ with relations

$$z^j z^k = \lambda^{jk} z^k z^j \ , \quad z^{j*} z^{k*} = \lambda^{jk} z^{k*} z^{j*}, \quad z^{j*} z^k = \lambda^{kj} z^{k*} z^{j*} \ , \tag{4.68}$$

with $j, k = 1, \ldots, n$. The $*$-algebra $\mathcal{A}(\mathbb{R}_\theta^{2n})$ can be thought of as the algebra of complex polynomials on the noncommutative $2n$-plane $\mathbb{R}_\theta^{2n}$ since it is a deformation of the commutative $*$-algebra $\mathcal{A}(\mathbb{R}^{2n})$ of complex polynomial functions on $\mathbb{R}^{2n}$ to which it reduces for $\theta = 0$. From relations (4.68), it follows that the elements $z^{j*} z^j = z^j z^{j*}\ ,\ j = 1, \ldots, n$, are in the center of $\mathcal{A}(\mathbb{R}_\theta^{2n})$. Since $\sum_{j=1}^n z^j z^{j*}$ is central as well, it makes sense to define $\mathcal{A}(S_\theta^{2n-1})$ to be the quotient of the $*$-algebra $\mathcal{A}(\mathbb{R}_\theta^{2n})$ by the ideal generated by $\sum_{j=1}^n z^j z^{j*} - 1$. The $*$-algebra $\mathcal{A}(S_\theta^{2n-1})$ can be thought of as the algebra of complex polynomials on the noncommutative $(2n-1)$-sphere S_θ^{2n-1} since it is a deformation of the commutative $*$-algebra $\mathcal{A}(S^{2n-1})$ of complex polynomial functions on the usual sphere S^{2n-1}.

Next, one defines $\mathcal{A}(\mathbb{R}_\theta^{2n+1})$ to be the complex unital $*$-algebra generated by $2n + 1$ elements made of $(z^j, z^{j*}, j = 1, \ldots, n)$ and of an addition hermitian element $x = x^*$ with relations like (4.68) and in addition

$$z^j x = x z^j \ , \quad j = 1, \ldots, n \ . \tag{4.69}$$

The $*$-algebra $\mathcal{A}(\mathbb{R}_\theta^{2n+1})$ is the algebra of complex polynomials on the noncommutative $(2n+1)$-plane $\mathbb{R}_\theta^{2n+1}$.

By the very definition, the elements $z^{j*} z^j = z^j z^{j*}\ ,\ j = 1, \ldots, n$, and x are in the center of $\mathcal{A}(\mathbb{R}_\theta^{2n+1})$ and so is the element $\sum_{j=1}^n z^j z^{j*} + x^2$. Then one defines $\mathcal{A}(S_\theta^{2n})$ to be the quotient of the $*$-algebra $\mathcal{A}(\mathbb{R}_\theta^{2n+1})$ by the ideal generated by $\sum_{j=1}^n z^j z^{j*} + x^2 - 1$. The $*$-algebra $\mathcal{A}(S_\theta^{2n})$ is the algebra of complex polynomials on the noncommutative $2n$-sphere S_θ^{2n} and is a deformation of the commutative $*$-algebra $\mathcal{A}(S^{2n})$ of complex polynomial functions on a usual sphere S^{2n}. By construction the sphere S_θ^{2n} is a suspension of the sphere S_θ^{2n-1}.

Next, let $\mathrm{Cliff}(\mathbb{R}_\theta^{2n})$ be the unital associative $*$-algebra over $\mathbb{C}$ generated by $2n$ elements $\Gamma^j, \Gamma^{k*}\ ,\ j, j = 1, \ldots, n$, with relations

$$\Gamma^j \Gamma^k + \lambda^{kj} \Gamma^k \Gamma^j = 0 \ ,$$
$$\Gamma^{j*} \Gamma^{k*} + \lambda^{kj} \Gamma^{k*} \Gamma^{j*} = 0 \ ,$$
$$\Gamma^j \Gamma^{k*} + \lambda^{jk} \Gamma^k \Gamma^{j*} = \delta^{jk} \mathbb{I} \ , \tag{4.70}$$

where $\mathbb{I}$ is the unit of the algebra and δ^{jk} is the usual flat metric. For $\theta = 0$ one gets the usual Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2n}$ of $\mathbb{R}^{2n}$. The element $\gamma \in \mathrm{Cliff}(\mathbb{R}_\theta^{2n})$ defined by

$$\gamma = \left[\Gamma^{1*}, \Gamma^1 \right] \cdot \ldots \cdot \left[\Gamma^{n*}, \Gamma^n \right] \tag{4.71}$$

is hermitian, $\gamma = \gamma^*$, satisfies

$$\gamma^2 = \mathbb{I}, \quad \gamma \Gamma^j + \Gamma^j \gamma = 0 \ , \quad \gamma \Gamma^{j*} + \Gamma^{j*} \gamma = 0 \ , \tag{4.72}$$

and determines a $\mathbb{Z}_2$-grading of $\mathrm{Cliff}(\mathbb{R}_\theta^{2n})$, $\Lambda \mapsto \gamma \Lambda \gamma$. In fact, one shows [15] that $\mathrm{Cliff}(\mathbb{R}_\theta^{2n})$ is isomorphic to the usual Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{2n}$ as a $*$-algebra and as a $\mathbb{Z}_2$-graded algebra. Furthermore, there is a representation of $\mathrm{Cliff}(\mathbb{R}_\theta^{2n})$ for which $\gamma = \left(\begin{smallmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{smallmatrix} \right)$

and $\Gamma^j \in \mathrm{Mat}_{2^n}(\mathbb{C})$ of the form

$$\Gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \overline{\sigma}^{j*} & 0 \end{pmatrix} \ , \qquad \Gamma^{j*} = \begin{pmatrix} 0 & \overline{\sigma}^{j*} \\ \sigma^j & 0 \end{pmatrix} \ , \tag{4.73}$$

with σ^j and $\overline{\sigma}^{j*}$ in $\mathrm{Mat}_{2^{n-1}}(\mathbb{C})$.

Theorem 4
a) *There is a canonical projection $e \in \mathrm{Mat}_{2^n}(\mathcal{A}(S_\theta^{2n}))$ given by*

$$e = \frac{1}{2} \left(\mathbb{I} + \sum_{j=1}^n (\Gamma^{j*} z^j + \Gamma^j z^{j*} + \gamma x) \right) \ , \tag{4.74}$$

where (z^j, z^{k*}, x) are the generators of $\mathcal{A}(S_\theta^{2n})$. Moreover, one has that

$$\mathrm{ch}_k(e) = 0 \ , \qquad 0 \le k \le n - 1 \, . \tag{4.75}$$

a) *There is a canonical unitary $u \in \mathrm{Mat}_{2^{n-1}}(\mathcal{A}(S_\theta^{2n-1}))$ given by*

$$u = \sum_{j+1}^n (\overline{\sigma}^j z^j + \sigma^j z^{j*}) \ , \tag{4.76}$$

where (z^j, z^{k*}) are the generators of $\mathcal{A}(S_\theta^{2n-1})$. Moreover, one has that

$$\mathrm{ch}_{k+\frac{1}{2}}(e) = 0 \ , \qquad 0 \le k \le n - 1 \, . \tag{4.77}$$

For a proof we refer to [15].
The projection e in (4.74) and the unitary u in (4.76) provide noncommutative solutions, via constraints (4.75) and (4.77), for the algebras $A_{m,r} = C(\mathrm{Gr}_{m,r})$ and $B_{m,r} = C(\mathrm{Gr}_{m,r})$ defined in (4.1). Thus, there are admissible surjections

$$A_{2n,2^n} \to C(S_\theta^{2n}) \ , \quad B_{2n-1,2^{n-1}} \to C(S_\theta^{2n-1}) \tag{4.78}$$

The projection (4.74) generalizes to higher dimensions the projection constructed in (4.10) for the four dimensional sphere S_θ.

4.8   Gauge theories

From Theorem 3 we know that the deformed spheres, and in particular the even ones, S_θ^{2n}, can be endowed with the structure of a noncommutative spin manifold, via a spectral triple $(C^\infty(S_\theta^{2n}), \mathcal{H}, D)$, which is isospectral since both the Hilbert space $\mathcal{H} = L^2(S^{2n}, \mathcal{S})$ and the Dirac operator D are the usual one on the commutative sphere S^{2n} whereas only the algebra and its representation on $\mathcal{H}$ are changed. In particular one could take the Dirac operator of the usual 'round' metric. Then out of this one can define a suitable Hodge operator $*_H$ on S_θ^{2n}. It turns out that the canonical projection (4.74) satisfies self-duality equations

$$*_H e(de)^n = \mathrm{i}^n e(de)^n. \tag{4.79}$$

These equations were somehow 'in the air'. For the four dimensional case I mentioned them during a talk in Ancona in February 2001 [41]. For the general case they were

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derived in [15] and also in [1]. Their 'commutative' counterparts were proposed in [29, 28] together with a description of gauge theories in terms of projectors.

In particular on the four dimensional sphere S_θ, one can develop Yang-Mills theory, since there are all the required structures, namely the algebra, the calculus (by means of the Dirac operator) and the "vector bundle" e.

The Yang-Mills action is given by,

$$YM(\nabla) = \oint \theta^2 \, ds^4 \, , \tag{4.80}$$

where $\theta = \nabla^2$ is the curvature, and $ds = D^{-1}$. This action has a strictly positive lower bound [10] given by a topological invariant which is just the index (3.28),

$$\varphi(e) = \oint \gamma(e - \frac{1}{2})[D, e]^4 \, ds^4 \, . \tag{4.81}$$

For the canonical projection (4.10), owing to (4.31), this topological invariant turns out to be just 1,

$$\varphi(e) = 1 \, . \tag{4.82}$$

An important problem, which is still open, is the construction and the classification of Yang-Mills connections in the noncommutative situation along the line of the ADHM construction [3]. For the noncommutative torus this was done in [19] and for a noncommutative $\mathbb{R}^4$ in [48].

5 Euclidean and Unitary Quantum Spheres

The contents of this Section is essentially a subset of the paper [34] by Eli Hawkins and myself.

The quantum Euclidean spheres in any dimensions, S_q^{N-1}, are (quantum) homogeneous spaces of quantum orthogonal groups, $\mathrm{SO}_q(N)$ [30]. The natural coaction of $\mathrm{SO}_q(N)$ on $\mathbb{R}_q^N$,

$$\delta : A(\mathbb{R}_q^N) \to A(\mathrm{SO}_q(N)) \otimes A(\mathbb{R}_q^N), \tag{5.1}$$

preserves the 'radius of the sphere' and yields a coaction of the quantum group $\mathrm{SO}_q(N)$ on S_q^{N-1}.

Similarly, 'odd dimensional' quantum spheres S_q^{2n-1} can be constructed as noncommutative homogeneous spaces of quantum unitary groups $\mathrm{SU}_q(n)$ [59] (see also [62]). Then, analogously to (5.1), there is also a coaction of the quantum group $\mathrm{SU}_q(n)$ on S_q^{2n-1}

$$\delta : A(S_q^{2n-1}) \to A(\mathrm{SU}_q(n)) \otimes A(S_q^{2n-1}). \tag{5.2}$$

In fact, it was realized in [34] that odd quantum Euclidean spheres are the same as unitary ones. This fact extends the classical result that odd dimensional spheres are simultaneously homogeneous spaces of orthogonal and of unitary groups.

The $*$-algebra $A(S_q^{N-1})$ of polynomial functions on each of the spheres S_q^{N-1} is given by generators and relations which were expressed in terms of a self-adjoint, unipotent matrix (a matrix of functions whose square is the identity) which is defined recursively. Instead in [42] the algebra was described by means of a suitable self-adjoint idempotent (a

matrix of functions whose square is itself). Let us then describe the algebra $A(S_q^{N-1})$. It is generated by elements $\{x_0 = x_0*, x_i, x_i^*, i = 1, \ldots, n\}$ for $N = 2n + 1$ while for $N = 2n$ there is no x_0. These generators obey the following commutation relations,

$$
\begin{aligned}
x_i x_j &= q x_j x_i, && 0 \leq i < j \leq n, \\
x_i^* x_j &= q x_j x_i^*, && i \neq j,
\end{aligned}
\tag{5.3a}
$$

$$
[x_i, x_i^*] = (1 - q^{-2}) s_{i-1},
\tag{5.3b}
$$

with the understanding that $x_0 = 0$ if $N = 2n$, so that in this case the generator x_1 is normal,

$$
x_1 x_1^* = x_1^* x_1 \qquad \text{in} \qquad A(S_q^{2n-1}).
\tag{5.4}
$$

The 'partial radii' $s_i \in A(S_q^{2n})$, are given recursively by

$$
\begin{aligned}
s_i &:= s_{i-1} + x_i^* x_i = q^{-2} s_{i-1} + x_i x_i^*, \\
s_0 &:= x_0^2,
\end{aligned}
\tag{5.5}
$$

and the last one s_n which can be shown to be central, is normalized to

$$
s_n = 1
\tag{5.6}
$$

We see that the equality of the two formulæ for the elements s_i in (5.5) is equivalent to the commutation relation (5.3b). These s_i are self-adjoint and related as

$$
0 \leq s_0 \leq \cdots \leq s_{n-1} \leq s_n = 1.
\tag{5.7}
$$

From the commutation relations (5.3a) it follows for $i < j$ that $x_i^* x_i x_j = q^2 x_j x_i^* x_i$; on the other hand $x_j^* x_j x_i = x_i x_j^* x_j$. By induction, we deduce that

$$
s_i x_j = \begin{cases} q^2 x_j s_i & : i < j \\ x_j s_i & : i \geq j, \end{cases}
\qquad
s_i x_j^* = \begin{cases} q^{-2} x_j^* s_i & : i < j \\ x_j^* s_i & : i \geq j \end{cases}
$$

and that the s_i's are mutually commuting. They can be used to construct representations of the algebra as we shall show later on.

As we have mentioned, in [42] it was shown that the defining relations of the algebra $A(S_q^{N-1})$ are equivalent to the condition that a certain matrix over $A(S_q^{N-1})$ be idempotent. In [34] it was proven that this is also equivalent to the condition that another matrix be unipotent, as we shall explain presently. First consider the even spheres S_q^{2n} for any integer $n > 0$. The algebra $A(S_q^{2n})$ is generated by elements $\{x_0, x_i, x_i^*, i = 1, \ldots, n\}$. Let us first consider the free unital $*$-algebra $F := \mathbb{C}\langle 1, x_0, x_i, x_i^*, i = 1, \ldots, n \rangle$ on $2n + 1$ generators. We recursively define self-adjoint matrices $u_{(2n)} \in \mathrm{Mat}_{2^n}(F)$ for all n by,

$$
u_{(2n)} := \begin{pmatrix} q^{-1} u_{(2n-2)} & x_n \\ x_n^* & -u_{(2n-2)} \end{pmatrix},
\tag{5.8}
$$

with $u_{(0)} = x_0$. The $*$-algebra $A(S_q^{2n})$ is then defined by the relations that $u_{(2n)}$ is unipotent, $u_{(2n)}^2 = 1$, and self-adjoint, $u_{(2n)}^* = u_{(2n)}$. That is, the algebra is the quotient of the free algebra F by these relations.

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The self-adjointness relations merely give that x_i^* is the adjoint of x_i and x_0 is self-adjoint. Unipotency gives a matrix of 2^{2n} relations, although many of these are vacuous or redundant. These can be deduced inductively from (5.8) which gives,

$$u_{(2n)}^2 = \begin{pmatrix} q^{-2}u_{(2n-2)}^2 + x_n x_n^* & q^{-1}u_{(2n-2)}x_n - x_n u_{(2n-2)} \\ q^{-1}x_n^* u_{(2n-2)} - u_{(2n-2)}x_n^* & u_{(2n-2)}^2 + x_n^* x_n \end{pmatrix}. \tag{5.9}$$

The condition that $u_{(2n)}^2 = 1$ means in particular that $u_{(2n)}^2$ is diagonal with all the diagonal entries equal. Looking at (5.9), we see that the same must be true of $u_{2n-2}^2 \in \mathrm{Mat}_{2^{n-1}}(A(S_q^{2n}))$, and so on. Thus, the diagonal relations require that all the diagonal entries of (each) $u_{(2j)}^2$ are equal. If this is true for $u_{(2j-2)}^2$, then the relation for $u_{(2j)}^2$ is that the same element (the diagonal entry) can be written in two different ways. This element is simply s_j and the two ways of writing it are those given in (5.5). Finally, $u_{(2n)}^2 = 1$ gives the relation $s_n = 1$. The off-diagonal relations are $q^{-1}u_{(2j-2)}x_j = x_j u_{(2j-2)}$ and $q^{-1}x_j^* u_{(2j-2)} = u_{(2j-2)}x_j^*$ for every $j = 1, \ldots, n$. Because the matrix $u_{(2j-2)}$ is constructed linearly from all of the generators x_i and x_i^* for $i < j$, this conditions are equivalent to the commutation relations (5.3a). This presentation of the relations by the unipotency of $u_{(2n)}$ is also the easiest way to see that there is an isomorphism $A(S_{1/q}^{2n}) \cong A(S_q^{2n})$ which is obtained by the substitutions $q \leftrightarrow q^{-1}$, $x_0 \to (-q)^n x_0$, and $x_i \to (-q)^{n-i}x_i^*$. This transforms the matrix $u_{(2n)} \to \widetilde{u}_{(2n)}$ and the latter is is unipotent and self-adjoint if and only if $u_{(2n)}$ is. Thus there is an isomorphism $A(S_{1/q}^{2n}) \cong A(S_q^{2n})$, and we can assume that $|q| > 1$ without loss of generality.

Next, consider the odd spheres S_q^{2n-1} for any integer $n > 0$. We can construct a unipotent $u_{(2n-1)} \in \mathrm{Mat}_{2^n}[A(S_q^{2n-1})]$, simply by setting $x_0 = 0$ in $u_{(2n)}$. Once again, the unipotency condition, $u_{(2n-1)}^2 = 1$, is equivalent to the relations defining the algebra $A(S_q^{2n-1})$ of polynomial functions on S_q^{2n-1}. Again, one defines self-adjoint elements $s_i \in A(S_q^{2n-1})$ such that $s_i = s_{i-1} + x_i^* x_i = q^{-2}s_{i-1} + x_i x_i^*$ with now $s_0 = x_0^2 = 0$. The commutation relations are again given by (5.3) but now (5.3b) gives in particular that the generator x_1 is normal, $x_1 x_1^* = x_1^* x_1$. The previous argument also shows that $A(S_q^{2n-1})$ is the quotient of $A(S_q^{2n})$ by the ideal generated by x_0. Geometrically, we may think of S_q^{2n-1} as a noncommutative subspace of S_q^{2n}. Because of the isomorphism $A(S_{1/q}^{2n}) \cong A(S_q^{2n})$, we have another isomorphism $A(S_{1/q}^{2n-1}) \cong A(S_q^{2n-1})$, and again we can assume that $|q| > 1$ without any loss of generality.

Remark 1 *The algebras of our spheres, both in even and odd 'dimensions', are generated by the entries of a projections. This is the same as the condition of full projection used by S. Waldmann in his analysis of Morita equivalence of star products [61].*

There is also a way of realizing even spheres as noncommutative subspaces of odd ones. Consider S_q^{2n+1}, set $x_1 = x_1^* = x_0$ and relabel x_2 as x_1, *et cetera*; let $u'_{(2n+1)}$ be the matrix obtained from $u_{(2n+1)}$ with these substitutions. The matrix $u'_{(2n+1)}$ is the same as $u_{(2n)}$ in which we substitute

$$x_0 \to \begin{pmatrix} 0 & x_0 \\ x_0 & 0 \end{pmatrix}, \qquad x_j \to \begin{pmatrix} x_j & 0 \\ 0 & x_j \end{pmatrix}, \quad j \neq 0.$$

Then the unipotency of $u'_{(2n+1)}$ yields precisely the same relations coming from the unipotency of $u_{(2n)}$. This shows that $A(S_q^{2n})$ is the quotient of $A(S_q^{2n+1})$ by the $*$-ideal generated by $x_1 - x_1^*$. Geometrically, we may think of S_q^{2n} as a noncommutative subspace of S_q^{2n+1}.

Summing up, every sphere contains a smaller sphere of dimension one less; by following this tower of inclusions to its base, we see that every sphere contains a classical S^1, because the circle does not deform. From this, it is easy to see that the spheres S_q^{N-1} have a S^1 worth of classical points. Indeed, with $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = 1$, there is a family of 1-dimensional representations (characters) of the algebra $A(S_q^{N-1})$ given by

$$\psi_\lambda(1) = 1, \qquad \psi_\lambda(x_n) = \lambda, \qquad \psi_\lambda((x_n)^*) = \overline{\lambda} ,$$
$$\psi_\lambda(x_i) = \psi_\lambda((x_i)^*) = 0 , \tag{5.10}$$

for $i = 0, 1, \ldots, n - 1$ or $i = 1, \ldots, n - 1$ according to whether $N = 2n + 1$ or $N = 2n$, respectively.

Each even sphere algebra has an involutive automorphism

$$\sigma : A(S_q^{2n}) \to A(S_q^{2n})$$
$$x_0 \mapsto -x_0; \qquad x_j \mapsto x_j, \quad j \neq 0, \tag{5.11}$$

which corresponds to flipping (reflecting) the classical S^{2n} across the hyperplane $x_0 = 0$. The coinvariant algebra of σ is the quotient of $A(S_q^{2n})$ by the ideal generated by x_0, which, as we have noted, is simply $A(S_q^{2n-1})$.
Geometrically this means that S_q^{2n-1} is the "equator" of S_q^{2n}, the subspace fixed by the flip.

As for odd spheres, they have an action $\rho : \mathbb{T} \to \operatorname{Aut}[A(S^{2n})]$ of the torus group $\mathbb{T}$, defined by multiplying x_1 by a phase and leaving the other generators unchanged,

$$\rho(\lambda) : A(S_q^{2n+1}) \to A(S_q^{2n+1})$$
$$x_1 \mapsto \lambda x_1; \qquad x_j \mapsto x_j, \quad j \neq 1. \tag{5.12}$$

The coinvariant algebra is given by setting $x_1 = 0$. Now, let $u''_{(2n+1)}$ be the matrix obtained by setting $x_1 = 0$ and relabeling x_2 as x_1, *et cetera*, in the matrix $u_{(2n+1)}$. Then, $u''_{(2n+1)}$ is equivalent to tensoring $u_{(2n-1)}$ with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$,

$$u''_{(2n+1)} = u_{(2n-1)} \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

and the result is unipotent if and only if $u_{(2n-1)}$ is; that is the unipotency of $u''_{(2n+1)}$ yields all and only the same relations coming from the unipotency of $u_{(2n-1)}$. This shows that $A(S_q^{2n-1})$ is the quotient of $A(S_q^{2n+1})$ by the $*$-ideal generated by x_1 and S_q^{2n-1} is the noncommutative subspace of S_q^{2n+1} fixed by the $\mathbb{T}$-action in (5.12).

5.1 The structure of the deformations

For each deformed sphere S_q^{N-1}, we have a one parameter family of algebras $A(S_q^{N-1})$ which, at $q = 1$, gives $A(S_1^{N-1}) = A(S^{N-1})$, the algebra of polynomial functions on a classical sphere S^{N-1}. It is possible to identify this one-parameter family of algebras to a fixed vector space and view the product as varying with the parameter: let us indicate this product with the symbol $*_q$. We can then construct a Poisson bracket on $A(S^{N-1})$ from the first derivative of the product at the classical parameter value, $q = 1$,

$$\{f, g\} := -i \left.\frac{d}{dq}\right|_{q=1} (f *_q g - g *_q f) . \tag{5.13}$$

The usual properties of a Poisson bracket (Leibniz and Jacobi identities) are simple consequences of associativity.

In general, given such a one-parameter deformation from a commutative manifold $\mathcal{M}$ into noncommutative algebras, we can construct a Poisson bracket on functions. This Poisson algebra, $A(\mathcal{M})$ with the commutative product and the Poisson bracket, describes the deformation to first order. A deformation is essentially a path through an enormous space of possible algebras, and the Poisson algebra is just a tangent. Nevertheless, if the deformation is well behaved the Poisson algebra does indicate where it is heading.

We shall also use the fact that the manifold $\mathbb{R}^{2n}$ has a unique symplectic structure, modulo isomorphism. Then, this symplectic structure corresponds to an essentially unique deformation. If we complete to a C*-algebra, then the deformation of $\mathcal{C}_0(\mathbb{R}^{2n})$ (continuous functions vanishing at infinity) will be the algebra, $\mathcal{K}$, of compact operators on a countably infinite-dimensional Hilbert space.

Let us go back to the spheres S_q^{N-1} and look more closely at them.

We have seen that the S_q^{2n-1} noncommutative subspace of S_q^{2n} corresponds to the equator, $S^{2n-1} \subset S^{2n}$, where $x_0 = 0$ and the Poisson structure on S^{2n} is degenerate. On the remaining $S^{2n} \smallsetminus S^{2n-1}$, the Poisson structure is nondegenerate. So, topologically, we have a union of two copies of symplectic $\mathbb{R}^{2n}$. Then, the kernel of the quotient map $A(S_q^{2n}) \to A(S_q^{2n-1})$ should be a deformation of the subalgebra of functions on S^{2n} which vanish at the equator. If we complete to C*-algebras, this should give us the direct sum of two copies of $\mathcal{K}$, one for each hemisphere. Thus we expect that the C*-algebra $\mathcal{C}(S_q^{2n})$ will be an extension:

$$0 \to \mathcal{K} \oplus \mathcal{K} \to \mathcal{C}(S_q^{2n}) \to \mathcal{C}(S_q^{2n-1}) \to 0. \tag{5.14}$$

In odd dimensions, the Poisson structure is necessarily degenerate. However, the S_q^{2n-1} noncommutative subspace of S_q^{2n+1} corresponds classically to the Poisson structure being more degenerate on $S^{2n-1} \subset S^{2n+1}$. It is of rank $2n$ at most points, but of rank $2n-2$ (or less) along S^{2n-1}. The complement $S^{2n+1} \smallsetminus S^{2n-1}$ has a symplectic foliation by $2n$ dimensional leaves which is invariant under the $\mathbb{T}$ action; the simplest possibility is that this corresponds to the product in the identification

$$S^{2n+1} \smallsetminus S^{2n-1} \cong S^1 \times \mathbb{R}^{2n}.$$

If we complete to C*-algebras, then the deformation of this should give the algebra $\mathcal{C}(S^1) \otimes \mathcal{K}$. The kernel of the quotient map $A(S_q^{2n+1}) \to A(S_q^{2n-1})$ should be this deformation, so we expect another extension,

$$0 \to \mathcal{C}(S^1) \otimes \mathcal{K} \to \mathcal{C}(S_q^{2n+1}) \to \mathcal{C}(S_q^{2n-1}) \to 0. \tag{5.15}$$

The extensions (5.14) and (5.15) turn out to be correct. As we have mentioned, the odd dimensional spheres we are considering are equivalent to the "unitary" odd quantum spheres of Vaksman and Soibelman [59]. In [36] Hong and Szymański obtained the C*-algebras $\mathcal{C}(S_q^{2n+1})$ as Cuntz-Krieger algebras of suitable graphs. From this construction they derived the extension (5.15). They also considered even spheres, defined as quotients of odd ones by the ideal generated by $x_1 - x_1^*$. These are thus isomorphic to the even spheres we are considering here. They also obtained these as Cuntz-Krieger algebras and derived the extension (5.14). However, as explicitly stated in the introduction to [36], they were unable to realize even spheres as quantum homogeneous spaces of quantum orthogonal groups, thus also failing to realize that "unitary" and "orthogonal" odd quantum spheres are the same.

5.2 Representations

We shall now exhibit all representations of the algebra $A(S_q^{N-1})$ which in turn extend to the C*-algebra $\mathcal{C}(S_q^{N-1})$.

Representations of the odd dimensional spheres were constructed in [59]. The primitive spectra of all these spheres were compute in [36], which amounts to a classification of representations. The representations for quantum Euclidean spheres have also been constructed in [31] by thinking of them as quotient algebras of quantum Euclidean planes.

The structure of the representations can be anticipated from the construction of S_q^{N-1} via the extensions (5.14) and (5.15) and by remembering that an irreducible representation ψ can be partially characterized by its kernel. Moreover, an irreducible representation of a C*-algebra restricts either to an irreducible or a trivial representation of any ideal; and conversely, an irreducible representation of an ideal extends to an irreducible representation of the C*-algebra (see for instance [26]).

For an even sphere S_q^{2n}, the kernel of an irreducible representation ψ will contain one or both of the copies of $\mathcal{K} \subset \mathcal{C}(S_q^{2n})$. If $\mathcal{K} \oplus \mathcal{K} \subseteq \ker \psi$, then ψ factors through $\mathcal{C}(S_q^{2n-1})$ and is given by a representation of that algebra. If one copy of $\mathcal{K}$ is not in $\ker \psi$, then ψ restricts to a representation of this $\mathcal{K}$. However, $\mathcal{K}$ has only one irreducible representation. Since $\mathcal{K}$ is an ideal in $\mathcal{C}(S_q^{2n})$, the unique irreducible representation of $\mathcal{K}$ uniquely extends to a representation of $\mathcal{C}(S_q^{2n})$ (with the other copy of $\mathcal{K}$ in its kernel).

Thus, up to isomorphism the irreducible representations of S_q^{2n} should be:

1. all irreducible representations of S_q^{2n-1},
2. a unique representation with kernel the second copy of $\mathcal{K}$,
3. a unique representation with kernel the first copy of $\mathcal{K}$.

From the extension (5.14) we expect that the generator x_0 is a self-adjoint element of $\mathcal{K} \oplus \mathcal{K} \subset \mathcal{C}(S_q^{2n})$ and it should have almost discrete, real spectrum: it will therefore be used to decompose the Hilbert space in a representation.

Similarly, from the construction of S_q^{2n+1} by the extension (5.15), one can anticipate the structure of its representations. Firstly, if $\mathcal{C}(S^1) \otimes \mathcal{K} \subseteq \ker \psi$, then ψ factors through $\mathcal{C}(S_q^{2n-1})$ and is really a representation of S_q^{2n-1}. Otherwise, ψ restricts to an irreducible representation of $\mathcal{C}(S^1) \otimes \mathcal{K}$. This factorizes as the tensor product of an irreducible representation of $\mathcal{C}(S^1)$ with one of $\mathcal{K}$. The irreducible representations of $\mathcal{C}(S^1)$ are simply given by the points of S^1, and as we have mentioned, $\mathcal{K}$ has a unique irreducible representation. The representations of $\mathcal{C}(S^1) \otimes \mathcal{K}$ are thus classified by the points of S^1. These representations extend uniquely from the ideal $\mathcal{C}(S^1) \otimes \mathcal{K}$ to the whole algebra $\mathcal{C}(S_q^{2n+1})$.

Thus, up to isomorphism, the irreducible representations of S_q^{2n+1} should be:

1. all irreducible representations of S_q^{2n-1},
2. a family of representations parameterized by S^1.

In the construction of the representations, a simple identity regarding the spectra of operators will be especially useful (see, for instance [54]). If x is an element of any C*-algebra, then

$$\{0\} \cup \operatorname{Spec} x^*x = \{0\} \cup \operatorname{Spec} xx^*. \tag{5.16}$$

5.3 Even sphere representations

To illustrate the general structure we shall start by describing the lowest dimensional case, namely S_q^2. This is isomorphic to the so-called equator sphere of Podleś [49]. For this sphere, the representations were also constructed in [46] in a way close to the one presented here.

Let us then consider the sphere S_q^2.

As we have discussed, we expect that, in some faithful representation, x_0 is a compact operator and thus has an almost discrete, real spectrum. However, we cannot assume *a priori* that x_0 has eigenvalues, let alone that its eigenvectors form a complete basis of the Hilbert space. The sphere relation $1 = x_0^2 + x_1^* x_1 = q^{-2} x_0^2 + x_1 x_1^*$ shows that $x_0^2 \leq 1$ and thus $\|x_0\| \leq 1$. As x_0 is self-adjoint, this shows that $\operatorname{Spec} x_0 \subseteq [-1, 1]$. By (5.16) we have also,

$$\{0\} \cup \operatorname{Spec} x_1^* x_1 = \{0\} \cup \operatorname{Spec} x_1 x_1^*$$
$$\{0\} \cup \operatorname{Spec}(1 - x_0^2) = \{0\} \cup \operatorname{Spec}(1 - q^{-2} x_0^2)$$
$$\{1\} \cup \operatorname{Spec} x_0^2 = \{1\} \cup q^{-2} \operatorname{Spec} x_0^2.$$

Because we have assumed that $|q| \geq 1$, the only subsets of $[0, 1]$ that satisfy this condition are $\{0\}$ and $\{0, q^{-2k} \mid k = 0, 1, \dots\}$.

If $x_0 \neq 0 \in \mathcal{C}(S_q^2)$ then $\operatorname{Spec} x_0^2$ is the latter set. We cannot simply assume that $x_0 \neq 0$, since not every $*$-algebra is a subalgebra of a C*-algebra; however, our explicit representations will show that that is the case here.

Now let $\mathcal{H}$ be a separable Hilbert space and suppose that we have an irreducible $*$-representation, $\psi : A(S_q^2) \to \mathcal{L}(\mathcal{H})$.

If $\psi(x_0) = 0$ then $1 = \psi(x_1)\psi(x_1)^* = \psi(x_1)^*\psi(x_1)$. Thus $\psi(x_1)$ is unitary, and by the assumption of irreducibility, it is a number $\lambda \in \mathbb{C}$, $|\lambda| = 1$. So, $\mathcal{H} = \mathbb{C}$ and the representation is $\psi_\lambda^{(1)}$ defined by,

$$\psi_\lambda^{(1)}(x_0) = 0; \qquad \psi_\lambda^{(1)}(x_1) = \lambda, \quad \lambda \in S^1. \tag{5.17}$$

Thus we have an S^1 worth of representations with x_0 in the kernel.

If $\psi(x_0) \neq 0$, then $1 \in \operatorname{Spec} x_0^2$; it is an isolated point in the spectrum and therefore an eigenvalue. For some sign $\pm$ there exists a unit vector $|0\rangle \in \mathcal{H}$ such that $\psi(x_0) |0\rangle = \pm |0\rangle$. The relation $x_0 x_1 = q x_1 x_0$ suggests that x_1 and x_1^* shift the eigenvalues of x_0. Indeed, for $k = 0, 1, \dots$, the vector $\psi(x_1^*)^k |0\rangle$ is an eigenvector as well, because

$$\psi(x_0)\psi(x_1^*)^k |0\rangle = q^{-k}\psi(x_1^{*k} x_0) |0\rangle = \pm q^{-k}\psi(x_1^*)^k |0\rangle.$$

By normalizing, we obtain a sequence of unit eigenvectors, defined by

$$|k\rangle := (1 - q^{-2k})^{-1/2}\psi(x_1^*) |k - 1\rangle.$$

We have thus two representations $\psi_+^{(2)}$ and $\psi_-^{(2)}$, and direct computation shows that

$$\psi_\pm^{(2)}(x_0) |k\rangle = \pm q^{-k} |k\rangle,$$
$$\psi_\pm^{(2)}(x_1) |k\rangle = (1 - q^{-2k})^{1/2} |k - 1\rangle, \tag{5.18}$$
$$\psi_\pm^{(2)}(x_1^*) |k\rangle = (1 - q^{-2(k+1)})^{1/2} |k + 1\rangle.$$

The eigenvectors $\{|k\rangle \mid k = 0, 1, \dots\}$ are mutually orthogonal because they have distinct eigenvalues, and by the assumption of irreducibility they form a basis for the Hilbert space $\mathcal{H}$.

Notice that any power of $\psi_\pm^{(2)}(x_0)$ is a trace class operator, while this is not the case for the operators $\psi_\pm^{(2)}(x_1)$ and $\psi_\pm^{(2)}(x_1^*)$ nor for any of their powers.

Note also that the representations (5.18) are related by the automorphism σ in (5.11), as

$$\psi_\pm^{(2)} \circ \sigma = \psi_\mp^{(2)}. \tag{5.19}$$

If we set a value of q with $|q| < 1$ in (5.18), the operators would be unbounded. This is the reason for assuming that $|q| > 1$. The assumption was used in computing $\operatorname{Spec} x_0$. Not only is $\|x_0\| \leq 1$, but by a similar calculation $\|x_0\| \leq |q|$. Which bound is more relevant obviously depends on whether q is greater or less than 1. For $|q| < 1$ the appropriate formulæ for the representations can be obtained from (5.18) by replacing the index k with $-k - 1$. As a consequence, the role of x_1 and x_1^* as lowering and raising operators is exchanged.

For the general even spheres S_q^{2n} the structure of the representations is similar to that for S_q^2, but more complicated. The element x_0 is no longer sufficient to completely decompose the Hilbert space of the representation and we need to use all the commuting self-adjoint elements $s_i \in A(S_q^{2n})$ defined in (5.5).

Suppose that $\psi : A(S_q^{2n}) \to \mathcal{L}(\mathcal{H})$ is an irreducible $*$-representation.

If $\psi(x_0) = 0$, then ψ factors through $A(S_q^{2n-1})$. Thus ψ is an irreducible representation of $A(S_q^{2n-1})$; these will be discussed later.

If $\psi(x_0) \neq 0$, then $\psi(s_0) \neq 0$, and by the relations (5.7), all of the $\psi(s_i)$'s are nonzero. Proceeding recursively, we find that there is a simultaneous eigenspace with eigenvalue 1 for all the $\psi(s_i)$'s. That is, there must exist a unit vector $|0, \dots, 0\rangle \in \mathcal{H}$ such that $\psi(s_i) |0, \dots, 0\rangle = |0, \dots, 0\rangle$ for all i and $\psi(x_0) |0, \dots, 0\rangle = \pm |0, \dots, 0\rangle$. More unit vectors are defined by
$$|k_0, \dots, k_{n-1}\rangle \sim \psi(x_1^*)^{k_0} \dots \psi(x_n^*)^{k_{n-1}} |0, \dots, 0\rangle$$
modulo a positive normalizing factor. Working out the correct normalizing factors we get two representations $\psi_\pm^{(2n)}$ defined by,

$$\psi_\pm^{(2n)}(x_0) |k_0, \dots, k_{n-1}\rangle = \pm q^{-(k_0 + \dots + k_{n-1})} |k_0, \dots, k_{n-1}\rangle \tag{5.20}$$
$$\psi_\pm^{(2n)}(x_i) |k_0, \dots, k_{n-1}\rangle = (1 - q^{-2k_{i-1}})^{1/2} q^{-(\sum_{j=i}^{n-1} k_j)} |\dots, k_{i-1} - 1, \dots\rangle$$
$$\psi_\pm^{(2n)}(x_i^*) |k_0, \dots, k_{n-1}\rangle = (1 - q^{-2(k_{i-1}+1)})^{1/2} q^{-(\sum_{j=i}^{n-1} k_j)} |\dots, k_{i-1} + 1, \dots\rangle$$

with $i = 1, \dots, n$. With this values for the index i, we see that x_i lowers k_{i-1}, whereas x_i^* raises k_{i-1}. From irreducibility the collection of vectors $\{|k_0, \dots, k_{n-1}\rangle, k_i \geq 0\}$ constitute a complete basis for the Hilbert space $\mathcal{H}$.

As before, the two representations (5.20) are related by the automorphism σ,

$$\psi_\pm^{(2n)} \circ \sigma = \psi_\mp^{(2n)}. \tag{5.21}$$

Again the formulæ (5.20) for the representations are corrected for $|q| > 1$; and again the representations for $|q| < 1$ can be obtained by replacing all indices k_i with $-k_i - 1$ in (5.20).

In all of the irreducible representations of $A(S_q^{2n})$, the representative of x_0 is compact; in fact it is trace class. We can deduce from this that the C*-ideal generated by $\psi_{\pm}^{(2n)}(x_0)$ in $\mathcal{C}(S_q^{2n})$ is isomorphic to $\mathcal{K}(\mathcal{H})$, the ideal of all compact operators on $\mathcal{H}$. By using the continuous functional calculus, we can apply any function $f \in \mathcal{C}[-1, 1]$ to x_0. If f is supported on $[0, 1]$, then $f(x_0) \in \ker \psi_{-}^{(2n)}$. Likewise if f is supported in $[-1, 0]$, then $f(x_0) \in \ker \psi_{+}^{(2n)}$. From this we deduce that the C*-ideal generated by x_0 in $\mathcal{C}(S_q^{2n})$ is $\mathcal{K} \oplus \mathcal{K}$. One copy of $\mathcal{K}$ is $\ker \psi_{+}^{(2n)}$; the other is $\ker \psi_{-}^{(2n)}$. Thus we get exactly the extension (5.14).

5.4   Odd sphere representations

Again, to illustrate the general strategy we shall work out in detail the simplest case, that of the sphere S_q^3. This can be identified with the underlying noncommutative space of the quantum group $\mathrm{SU}_q(2)$ and as such the representations of the algebra are well known [64].

The generators $\{x_i, x_i^* \mid i = 1, 2\}$ of the algebra $A(S_q^3)$ satisfy the commutation relations $x_1 x_2 = q x_2 x_1$, $x_i^* x_j = q x_j x_i^*, i \neq j$, $[x_1, x_1^*] = 0$, and $[x_2, x_2^*] = (1 - q^{-2}) x_1 x_1^*$. Furthermore, there is also the sphere relation $1 = x_2^* x_2 + x_1^* x_1 = x_2 x_2^* + q^{-2} x_1 x_1^*$. The normal generator x_1 plays much the same role for the representations of S_q^3 that x_0 does for those of S_q^2. The sphere relation shows that $\|x_1\| \leq 1$ and

$$\{0\} \cup \mathrm{Spec}\, x_2^* x_2 = \{0\} \cup \mathrm{Spec}\, x_2 x_2^*$$
$$\{0\} \cup \mathrm{Spec}(1 - x_1^* x_1) = \{0\} \cup \mathrm{Spec}(1 - q^{-2} x_1 x_1^*)$$
$$\{1\} \cup \mathrm{Spec}\, x_1^* x_1 = \{1\} \cup q^{-2}\, \mathrm{Spec}\, x_1^* x_1,$$

which shows that either $x_1 = 0$ or $\mathrm{Spec}\, x_1^* x_1 = \{0, q^{-2k} \mid k = 0, 1, \dots\}$.

Let $\psi : A(S_q^3) \to \mathcal{L}(\mathcal{H})$ be an irreducible $*$-representation. If $\psi(x_1) = 0$ then the relations reduce to $1 = \psi(x_2)\psi(x_2)^* = \psi(x_2)^*\psi(x_2)$. Thus $\psi(x_2)$ is unitary and by the assumption of irreducibility, it is a scalar, $\psi(x_2) = \lambda \in \mathbb{C}$ with $|\lambda| = 1$. Thus, as before, we have an S^1 of representations of this kind. If $\psi(x_1) \neq 0$, then $1 \in \mathrm{Spec}\, \psi(x_1^* x_1)$ and is an isolated point in the spectrum. Thus, there exists a unit vector $|0\rangle \in \mathcal{H}$ such that $\psi(x_1^* x_1)|0\rangle = |0\rangle$, and by the assumption of irreducibility, there is some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\psi(x_1)|0\rangle = \lambda|0\rangle$. We see then that $\psi(x_2^*)^k |0\rangle$ is an eigenvector

$$\psi(x_1)\psi(x_2^*)^k |0\rangle = q^{-k}\psi(x_2^{*k} x_1)|0\rangle = \lambda q^{-k}\psi(x_2^*)^k |0\rangle.$$

By normalizing, we get a sequence of unit eigenvectors recursively defined by

$$|k\rangle := (1 - q^{-2k})^{-1/2}\psi(x_2^*)|k - 1\rangle.$$

A family of representations $\psi_\lambda^{(3)}$, $\lambda \in S^1$, is then defined by

$$\psi_\lambda^{(3)}(x_1)|k\rangle = \lambda q^{-k}|k\rangle,$$
$$\psi_\lambda^{(3)}(x_1^*)|k\rangle = \overline{\lambda} q^{-k}|k\rangle,$$
$$\psi_\lambda^{(3)}(x_2)|k\rangle = (1 - q^{-2k})^{1/2}|k - 1\rangle,$$
$$\psi_\lambda^{(3)}(x_2^*)|k\rangle = (1 - q^{-2(k+1)})^{1/2}|k + 1\rangle. \tag{5.22}$$

We notice that any power of $\psi_\lambda^{(3)}(x_1)$ or $\psi_\lambda^{(3)}(x_1^*)$ is a trace class operator, while this is not the case for the operators $\psi_\lambda^{(3)}(x_2)$ and $\psi_\lambda^{(3)}(x_2^*)$ nor for any of their powers.

Next, for the general odd spheres S_q^{2n+1} , let $\psi : A(S_q^{2n+1}) \to \mathcal{L}(\mathcal{H})$ be an irreducible representation.

If $\psi(x_1) = 0$ then ψ factors through $A(S_q^{2n-1})$ and is an irreducible representation of that algebra.

If $\psi(x_1) \neq 0$ then $\psi(s_1) \neq 0, \psi(s_2) \neq 0$, *et cetera*. By the same arguments as for S_q^{2n}, there must exist a simultaneous eigenspace with eigenvalue 1 for all of $s_1, \ldots s_n$. By the assumption of irreducibility, this eigenspace is 1-dimensional. Let $|0, \ldots, 0\rangle \in \mathcal{H}$ be a unit vector in this eigenspace. Then $s_i |0, \ldots, 0\rangle = |0, \ldots, 0\rangle$ for $i = 1, \ldots, n$. The restriction of $\psi(x_1)$ to this subspace is unitary and thus for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, one has that $\psi(x_1) |0, \ldots, 0\rangle = \lambda |0, \ldots, 0\rangle$. We can construct more simultaneous eigenvectors of the s_i's by defining

$$|k_1, \ldots, k_n\rangle \sim \psi(x_2)^{k_1} \ldots \psi(x_{n+1})^{k_n} |0, \ldots, 0\rangle$$

modulo a positive normalizing constant. Working out the normalization, one has a family of representations $\psi_\lambda^{(2n+1)}$,

$$
\begin{aligned}
\psi_\lambda^{(2n+1)}(x_1) |k_1, \ldots, k_n\rangle &= \lambda q^{-(k_1 + \cdots + k_n)} |k_1, \ldots, k_n\rangle, \\
\psi_\lambda^{(2n+1)}(x_1^*) |k_1, \ldots, k_n\rangle &= \overline{\lambda} q^{-(k_1 + \cdots + k_n)} |k_1, \ldots, k_n\rangle, \\
\psi_\lambda^{(2n+1)}(x_i) |k_1, \ldots, k_n\rangle &= (1 - q^{-2k_{i-1}})^{1/2} q^{-(\sum_{j=i}^n k_j)} |\ldots, k_{i-1} - 1, \ldots\rangle, \\
\psi_\lambda^{(2n+1)}(x_i^*) |k_1, \ldots, k_n\rangle &= (1 - q^{-2(k_{i-1}+1)})^{1/2} q^{-(\sum_{j=i}^n k_j)} |\ldots, k_{i-1} + 1, \ldots\rangle,
\end{aligned}
\tag{5.23}
$$

for $i = 2, \ldots, n + 1$. With this values for the index i, x_i lowers k_{i-1}, whereas x_i^* raises k_{i-1}. From irreducibility the vectors $\{|k_1, \ldots, k_n\rangle, k_i \geq 0\}$ form an orthonormal basis of the Hilbert space $\mathcal{H}$.

As for the even case, the formulæ (5.23) give bounded operators only for $|q| > 1$; and as before, the representations for $|q| < 1$ can be obtained by replacing all indices k_i with $-k_i - 1$.

Again, as in the even case, we can verify that $\psi_\lambda^{(2n+1)}(x_1)$ is compact (indeed, trace class) and that the ideal generated by $\psi_\lambda^{(2n+1)}(x_1)$ is $\mathcal{K}(\mathcal{H})$, in the C*-algebra completion of the image $\psi_\lambda^{(2n+1)}(A(S_q^{2n+1}))$. The representations $\psi_\lambda^{(2n+1)}$ can be assembled into a single representation by adjointable operators on a Hilbert $\mathcal{C}(S^1)$-module. With this we can verify that the ideal generated by x_1 in $\mathcal{C}(S_q^{2n+1})$ is $\mathcal{C}(S^1) \otimes \mathcal{K}$ and this verifies the extension (5.15).

Summing up, we get a complete picture of the set of irreducible representations of all these spheres S_q^N; or equivalently, of the primitive spectrum of the C*-algebra $\mathcal{C}(S_q^N)$ of continuous functions on S_q^N.

For the odd spheres S_q^{2n+1}, the set of irreducible representations is indexed by the union of $n + 1$ copies of S^1. These run from the representations $\psi_\lambda^{(2n+1)}$ of S_q^{2n+1} given in (5.23) down to the one dimensional representations $\psi_\lambda^{(1)}$ that factor through the undeformed S^1. For the even spheres S_q^{2n}, the set of irreducible representations is indexed by the union of n copies of S^1 and 2 points. The isolated points correspond to the 2 representations $\psi_\pm^{(2n)}$ specific to S_q^{2n} and given in (5.20); the circles correspond to representations $\psi_\lambda^{(2m+1)}$ coming from lower odd dimensional spheres, down to the undeformed S^1.

6 K-homology and K-theory for Quantum Spheres

We explicitly construct complete sets of generators for the K-theory (by nontrivial self-adjoint idempotents and unitaries) and the K-homology (by nontrivial Fredholm modules) of the spheres S_q^{N-1}. We also construct the corresponding Chern characters in cyclic homology and cohomology and compute the pairing of K-theory with K-homology.

   We shall study generators of the K-homology and K-theory of the spheres S_q^{N-1}. The K-theory classes will be given by means of self-adjoint idempotents (naturally associated with the aforementioned unipotents) and of unitaries in algebras of matrices over $A(S_q^{N-1})$. The K-homology classes will be given as (homotopy classes of) suitable 1-summable Fredholm modules using the representations constructed previously.

For odd spheres (i.e. for N even) the odd K-homology generators are first given in terms of unbounded Fredholm modules. These are given by means of a natural unbounded operator D which, while failing to have compact resolvent, has bounded commutators with all elements in the algebra $A(S_q^{2n-1})$.

   In fact, in order to compute the pairing of K-theory with K-homology, it is more convenient to first compute the Chern characters and then use the pairing between cyclic homology and cohomology [10]. Thus, together with the generators of K-theory and K-homology we shall also construct the associated Chern characters in the cyclic homology $HC_*[A(S_q^{N-1})]$ and cyclic cohomology $HC^*[A(S_q^{N-1})]$ respectively.

   Needless to say, the pairing is integral (it comes from a noncommutative index theorem). The non-vanishing of the pairing will testify to the non-triviality of the elements that we construct in both K-homology K-theory.

   It is worth recalling the K-theory and homology of the classical spheres.
For an even dimensional sphere S^{2n}, the groups are

$$K^0(S^{2n}) \cong \mathbb{Z}^2, \qquad K^1(S^{2n}) = 0,$$
$$K_0(S^{2n}) \cong \mathbb{Z}^2, \qquad K_1(S^{2n}) = 0.$$

One generator of the K-theory $[1] \in K^0(S^{2n})$ is given by the trivial 1-dimensional bundle. The other generator of $K^0(S^{2n})$ is the left handed spinor bundle. One K-homology generator $[\varepsilon] \in K_0(S^{2n})$ is "trivial" and is the push-forward of the generator of $K_0(*) \cong \mathbb{Z}$ by the inclusion $\iota : * \hookrightarrow S^{2n}$ of a point (any point) into the sphere. The other generator, $[\mu] \in K_0(S^{2n})$, is the K-orientation of S^{2n} given by its structure as a spin manifold [10].

   For an odd dimensional sphere, the groups are

$$K^0(S^{2n+1}) \cong \mathbb{Z}, \qquad K^1(S^{2n+1}) \cong \mathbb{Z},$$
$$K_0(S^{2n+1}) \cong \mathbb{Z}, \qquad K_1(S^{2n+1}) \cong \mathbb{Z}.$$

The generator $[1] \in K^0(S^{2n+1})$ is the trivial 1-dimensional bundle. The generator of $K^1(S^{2n+1})$ is a nontrivial unitary matrix-valued function on S^{2n+1}; for instance, it may be takes as the matrix (6.13) in the limit $q = 1$. The trivial generator $[\varepsilon] \in K_0(S^{2n+1})$ is again given by the inclusion of a point. The generator $[\mu] \in K_1(S^{2n+1})$ is the K-orientation of S^{2n+1} given by its structure as a spin manifold [10].

   There is a natural pairing between K-homology and K-theory. If we pair $[\varepsilon]$ with a vector bundle we get the rank of the vector bundle, i. e. the dimension of its fibers. If we pair $[\mu]$ with a vector bundle it gives the "degree" of the bundle, a measure of its nontriviality. Similarly, pairing with $[\mu]$ measures the nontriviality of a unitary.

The K-theory and K-homology of the quantum Euclidean spheres are isomorphic to that of the classical spheres; that is, for any N and q, one has that $K_*[\mathcal{C}(S_q^{N-1})] \cong K^*(S^{N-1})$ and $K^*[\mathcal{C}(S_q^{N-1})] \cong K_*(S^{N-1})$. In the case of K-theory, this was proven by Hong and Szymański in [36] using their construction of the C*-algebras as Cuntz-Krieger algebras of graphs. The groups K_0 and K_1 were given as the cokernel and the kernel respectively, of a matrix canonically associated with the graph. The result for K-homology can be proven using the same techniques [21, 50]: the groups K^0 and K^1 are now given as the kernel and the cokernel respectively, of the transposed matrix. The K-theory and the K-homology for the particular case of S_q^2 (in fact for all Podleś spheres S_{qc}^2) was worked out in [46]) while for $S^3 \cong SU_q(2)$ it was spelled out in [45].

6.1 K-homology

Because the K-homology of these deformed spheres is isomorphic to the K-homology of the ordinary spheres, we need to construct two independent generators. First consider the "trivial" generator of $K^0[\mathcal{C}(S_q^{N-1})]$. This can be constructed in a manner closely analogous to the undeformed case.

As we have already mentioned, the trivial generator of $K_0(S^{N-1})$ is the image of the generator of the K-homology of a point by the functorial map $K_*(\iota) : K_0(*) \to K_0(S^{N-1})$, where $\iota : * \hookrightarrow S^{N-1}$ is the inclusion of a point into the sphere. The quantum Euclidean spheres do not have as many points, but they do have some. We have seen that the relations among the various spheres always include a homomorphism $A(S_q^{N-1}) \to A(S^1)$. Equivalently, every S_q^{N-1} has a circle S^1 as a classical subspace; thus for every $\lambda \in S^1$ there is a point, i. e., the homomorphism $\psi_\lambda^{(1)} : \mathcal{C}(S_q^{N-1}) \to \mathbb{C}$.

We can construct an element $[\varepsilon_\lambda] \in K^0[\mathcal{C}(S_q^{N-1})]$ by pulling back the generator of $K^0(\mathbb{C})$ by $\psi_\lambda^{(1)}$. This construction factors through $K_0(S^1)$. Because S^1 is path connected, the points of S^1 all define homotopic (and hence K-homologous) Fredholm modules. Thus there is a single K-homology class $[\varepsilon_\lambda] \in K^0[\mathcal{C}(S_q^{N-1})]$, independent of $\lambda \in S^1$.

The canonical generator of $K^0(\mathbb{C})$ is given by the following Fredholm module: The Hilbert space is $\mathbb{C}$; the grading operator is $\gamma = 1$; the representation is the obvious representation of $\mathbb{C}$ on $\mathbb{C}$; the Fredholm operator is 0. If we pull this back to $K^0[\mathcal{C}(S_q^{N-1})]$ using $\psi_\lambda^{(1)}$, then the Fredholm module ε_λ is given in the same way but with $\psi_\lambda^{(1)}$ for the representation.

Given this construction of ε_λ, it is straightforward to compute its Chern character $\mathrm{ch}^*(\varepsilon_\lambda) \in HC^*[A(S_q^{N-1})]$: It is the pull back of the Chern character of the canonical generator of $K^0(\mathbb{C})$. An element of the cyclic cohomology HC^0 is a trace. The degree 0 part of the Chern character of the canonical generator of $K^0(\mathbb{C})$ is given by the identity map $\mathbb{C} \to \mathbb{C}$, which is trivially a trace. Pulling this back we find $\mathrm{ch}^0(\varepsilon_\lambda) = \psi_\lambda^{(1)} : A(S_q^{N-1}) \to \mathbb{C}$ which is also a trace because it is a homomorphism to a commutative algebra. These are distinct elements of $HC^0[A(S_q^{N-1})]$ for different values of λ. However, because the Fredholm modules ε_λ all lie in the same K-homology class, their Chern characters are all equivalent in *periodic* cyclic cohomology defined in (2.21). Indeed, applying the periodicity operator (2.20) once one gets that the cohomology classes $S(\psi_\lambda^{(1)}) \in HC^2[A(S_q^{N-1})]$ are all the same. For the computation of the pairing between K-theory and K-homology, any trace determining the same periodic cyclic cohomology class can be used. The most

symmetric choice of trace is given by averaging $\psi_\lambda^{(1)}$ over $\lambda \in S^1 \subset \mathbb{C}$:

$$\tau^0(a) := \oint_{S^1} \psi_\lambda^{(1)}(a) \frac{d\lambda}{2\pi i \lambda}.$$

The result is normalized, $\tau^0(1) = 1$, and vanishes on all the generators. The higher degree parts of $\mathrm{ch}^*(\varepsilon_\lambda)$ depend only on the K-homology class $[\varepsilon_\lambda]$ and can be constructed from τ^0 by the periodicity operator (2.20).

6.2 Fredholm Modules for even spheres

We will now construct an element $[\mu_{\mathrm{ev}}] \in K^0[\mathcal{C}(S_q^{2n})]$ by giving a suitable even Fredholm module $\mu := (\mathcal{H}, F, \gamma)$.

Identify the Hilbert spaces for the representations $\psi_\pm^{(2n)}$ given in (5.20) by identifying their bases, and call this $\mathcal{H}$. The representation for the Fredholm module is

$$\psi := \psi_+^{(2n)} \oplus \psi_-^{(2n)}$$

acting on $\mathcal{H} \oplus \mathcal{H}$. The grading operator and the Fredholm operator are respectively,

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is obvious that F is odd (since it anticommutes with γ) and Fredholm (since it is invertible). The remaining property to check is that for any $a \in A(S_q^{2n})$, the commutator $[F, \psi(a)]_-$ is compact. Indeed,

$$[F, \psi(a)]_- = \begin{pmatrix} 0 & -\psi_+^{(2n)}(a) + \psi_-^{(2n)}(a) \\ \psi_+^{(2n)}(a) - \psi_-^{(2n)}(a) & 0 \end{pmatrix}.$$

However, $\psi_+^{(2n)}(a) - \psi_-^{(2n)}(a) = \psi_+^{(2n)}[a - \sigma(a)]$ and $a - \sigma(a)$ is always proportional to a power of x_0. Thus this is not only compact, it is trace class. This also shows that we have (at least) a 1-summable Fredholm module. This is in contrast to the fact that the analogous element of $K_0(S^{2n})$ for the undeformed sphere is given by a $2n$-summable Fredholm module.

The corresponding Chern character [10] $\mathrm{ch}^*(\mu_{\mathrm{ev}})$ has a component in degree 0, $\mathrm{ch}^0(\mu_{\mathrm{ev}}) \in HC^0[A(S_q^{2n})]$. From the general construction (3.21), the element $\mathrm{ch}^0(\mu_{\mathrm{ev}})$ is the trace

$$\tau^1(a) := \tfrac{1}{2} \mathrm{Tr}\left(\gamma F([F, \psi(a)])\right) = \mathrm{Tr}\left[\psi_+^{(2n)}(a) - \psi_-^{(2n)}(a)\right]. \tag{6.1}$$

As we have mentioned, $\psi_+^{(2n)}(a) - \psi_-^{(2n)}(a) = \psi_+^{(2n)}[a - \sigma(a)]$ is trace class since $a - \sigma(a)$ is always proportional to a power of x_0. The higher degree parts of $\mathrm{ch}^*(\mu_{\mathrm{ev}})$ can be obtained via the periodicity operator (2.20).

For S_q^2 our Fredholm module coincides with the one constructed in [46].

6.3 Fredholm Modules for odd spheres

The element $[\mu_{\mathrm{odd}}] \in K^1[\mathcal{C}(S_q^{2n+1})]$ is most easily given by an unbounded Fredholm module. The corresponding unbounded operator D which, while failing to have compact resolvent, has bounded commutators with all elements in the algebra $A(S_q^{N-1})$.

Let the representation ψ be the direct integral (over $\lambda \in S^1$) of the representations $\psi_\lambda^{(2n+1)}$ given in (5.23). The operator is the unbounded 'Dirac' operator $D := \lambda^{-1}\frac{d}{d\lambda}$.

From (5.23), we see that the representative of x_1 is proportional to λ and as a consequence,

$$[D, \psi(x_1)]_- = \psi(x_1) \tag{6.2a}$$

whereas for $i > 1$ the representative of x_i does not involve λ and therefore

$$[D, \psi(x_i)]_- = 0, \quad i > 0. \tag{6.2b}$$

Since $a \mapsto [D, \psi(a)]_-$ is a derivation, this shows that $[D, \psi(a)]_-$ is bounded for any $a \in A(S_q^{2n+1})$. Note however that for $n > 0$ (i. e., except for S^1) all eigenvalues of D have infinite degeneracy and therefore D does not have compact resolvent.

This triple can be converted in to a bounded Fredholm module by applying a cutoff function to D. A convenient choice is $F = \chi(D)$ where

$$\chi(m) := \begin{cases} 1 & : m > 0 \\ -1 & : m \leq 0. \end{cases}$$

To be more explicit, use a Fourier series basis for the Hilbert space,

$$|k_0, k_1, \ldots, k_n\rangle := \lambda^{k_0} |k_1, \ldots, k_n\rangle,$$

in which the representation is given by,

$$\psi(x_1) |k_0, \ldots, k_n\rangle = q^{-(k_1+\cdots+k_n)} |k_0 + 1, \ldots, k_n\rangle,$$
$$\psi(x_1^*) |k_0, \ldots, k_n\rangle = q^{-(k_1+\cdots+k_n)} |k_0 - 1, \ldots, k_n\rangle,$$
$$\psi(x_i) |k_0, \ldots, k_n\rangle = (1 - q^{-2k_{i-1}})^{1/2} q^{-(k_i+\cdots+k_n)} |\ldots, k_{i-1} - 1, \ldots\rangle,$$
$$\psi(x_i^*) |k_0, \ldots, k_n\rangle = (1 - q^{-2(k_{i-1}+1)})^{1/2} q^{-(\sum_{j=i}^{n} k_j)} |\ldots, k_{i-1} + 1, \ldots\rangle,$$

for $i = 1, \ldots, n$. The Fredholm operator is then given by

$$F |k_0, \ldots, k_n\rangle = \chi(k_0) |k_0, \ldots, k_n\rangle.$$

The only condition to check is that the commutator $[F, \psi(a)]_-$ is compact for any $a \in \mathcal{C}(S_q^{2n+1})$. Since $a \mapsto [F, \psi(a)]_-$ is a derivation, it is sufficient to check this on generators. One finds

$$[F, \psi(x_i)]_- = 0 , \quad i > 1 ,$$
$$[F, \psi(x_1)]_- |k_0, \ldots, k_n\rangle = \begin{cases} 2q^{-(k_1+\cdots+k_n)} |1, k_1, \ldots, k_n\rangle & : k_0 = 0 \\ 0 & : k_0 \neq 0, \end{cases} \tag{6.3}$$

which is indeed compact, and in fact trace class.

Thus, this is a 1-summable Fredholm module. Again this is in contrast to the fact that

the analogous element of $K_1(S^{2n+1})$ for the undeformed sphere is given by a $(2n+1)$-summable Fredholm module.

Its Chern character [10] begins with $\mathrm{ch}^{\frac{1}{2}}(\mu_{\mathrm{odd}}) \in HC^1[A(S_q^{2n+1})]$. From the general construction (3.20), the element $\mathrm{ch}^{\frac{1}{2}}(\mu_{\mathrm{odd}})$ is given by the cyclic 1-cocycle φ defined by

$$\varphi(a, b) := \tfrac{1}{2} \mathrm{Tr}\left(\psi(a)[F, \psi(b)]_-\right). \tag{6.4}$$

One checks directly cyclicity, i.e. $\varphi(a, b) = -\varphi(b, a)$, and closure under b, i.e. $\varphi(ab, c) - \varphi(a, bc) + \varphi(ca, b) = b\varphi(a, b, c)$.

The higher degree parts of $\mathrm{ch}^*(\mu_{\mathrm{odd}})$ can be obtained via the periodicity operator (2.20).

For $S_q^3 \cong \mathrm{SU}_q(2)$ our Fredholm module coincides with the one in [45].

6.4 Singular integrals

We could interpret the classes $[\mu_{\mathrm{ev}}] \in K^0[\mathcal{C}(S_q^{2n})]$ and $[\mu_{\mathrm{odd}}] \in K^1[\mathcal{C}(S_q^{2n+1})]$ as giving 'singular' integrals over the corresponding quantum spheres. With the associated Chern characters given in (6.1) and (6.4) respectively, and from the general expression (3.19), these integrals are given by,

$$\int_{S_q^{2n}} a = \tau^1(a), \quad \forall a \in \mathcal{A}(S_q^{2n}), \tag{6.5}$$

$$\int_{S_q^{2n+1}} a\, db = \phi(a, b), \quad \forall a, b \in \mathcal{A}(S_q^{2n+1}). \tag{6.6}$$

As a way of illustration, let us compute them on generators. We indicate with δ_{ij} the usual Kronecker delta which is equal to 1 if $i = j$ and 0 otherwise.

Firstly, for even spheres we find

$$\int_{S_q^{2n}} x_i = \tau^1(x_i) = \mathrm{Tr}\left[\psi_+^{(2n)}\left(x_i - \sigma(x_i)\right)\right] = 2\,\mathrm{Tr}\left[\psi_+^{(2n)}(x_i)\right]\delta_{i0}. \tag{6.7}$$

Thus, we need to compute

$$\mathrm{Tr}[\psi_+^{(2n)}(x_0)] = \sum_{k_0=0}^{\infty} \cdots \sum_{k_{n-1}=0}^{\infty} q^{-(k_0 + \cdots + k_{n-1})} = \left(\sum_{k=0}^{\infty} q^{-k}\right)^n$$

$$= (1 - q^{-1})^{-n},$$

and in turn,

$$\int_{S_q^{2n}} x_i = \frac{2}{(1 - q^{-1})^n}\,\delta_{i0}. \tag{6.8}$$

Similarly, for odd spheres we find

$$\int_{S_q^{2n+1}} x_i\, dx_j^* = \phi(x_i, x_i^*) = \frac{1}{2}\,\mathrm{Tr}\left(\psi(x_1^*)[F, \psi(x_1)]_-\right)\delta_{i1}\delta_{j1}. \tag{6.9}$$

We have already computed $[F, \psi(x_1)]_-$ in eq. (6.3). From that, we get

$$\psi(x_1^*)[F, \psi(x_1)]_- |k_0, \ldots, k_n\rangle = \begin{cases} 2q^{-2(k_1 + \cdots + k_n)} |0, k_1, \ldots, k_n\rangle & : k_0 = 0 \\ 0 & : k_0 \neq 0. \end{cases}$$

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Thus,

$$\mathrm{Tr}\,(\psi(x_1^*)[F,\psi(x_1)]_-) = \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} 2q^{-2(k_1+\cdots+k_n)} = 2\left(\sum_{k=0}^{\infty}q^{-2k}\right)^n$$
$$= 2(1-q^{-2})^{-n},$$

and in turn,

$$\int_{S_q^{2n+1}} x_i\,dx_j^* = \frac{1}{(1-q^{-2})^n}\,\delta_{i1}\delta_{j1}\,. \tag{6.10}$$

6.5 K-theory for even spheres

For S_q^{2n} we construct two classes in the K-theory group $K_0[\mathcal{C}(S_q^{2n})]\cong\mathbb{Z}^2$.
The first class is trivial. The element $[1]\in K_0[\mathcal{C}(S_q^{2n})]$ is the equivalence class of $1\in\mathcal{C}(S_q^{2n})$ which is of course an idempotent. In order to compute the pairing with K-homology, we need the degree 0 part of its Chern character, $\mathrm{ch}_0[1]$, which is represented by the cyclic cycle 1.
The second, nontrivial, class was presented in [42]. It is given by an idempotent $e_{(2n)}$ constructed from the unipotent (5.8) as

$$e_{(2n)} = \tfrac{1}{2}(\mathbb{I} + u_{(2n)}) \tag{6.11}$$

(again, for the sphere S_q^2 the idempotent (6.11) was already in [46]). Its degree 0 Chern character, $\mathrm{ch}_0(e_{(2n)})\in \hat{HC}_0[A(S_q^{2n})]$, is

$$\mathrm{ch}_0(e_{(2n)}) = \mathrm{tr}(e_{(2n)} - \tfrac{1}{2}\mathbb{I}_{2n}) = \tfrac{1}{2}\,\mathrm{tr}(u_{(2n)})$$
$$= \tfrac{1}{2}(q^{-1}-1)^n x_0, \tag{6.12}$$

since the recursive definition (5.8) of the unipotent $u_{(2n)}$ shows that,

$$\mathrm{tr}(u_{(2n)}) = (q^{-1}-1)\,\mathrm{tr}(u_{(2n-2)}) = (q^{-1}-1)^n x_0.$$

Now, we can pair these classes with the two K-homology elements which we constructed in Section 6.1. First,

$$\langle\varepsilon_\lambda,[1]\rangle := \tau^0(1) = 1,$$

which is hardly surprising. Second, the "rank" of the idempotent $e_{(2n)}$ is

$$\langle\varepsilon_\lambda, e_{(2n)}\rangle := \tau^0(\mathrm{tr}(e_{(2n)}) = 2^{n-1}.$$

Also not surprising is the "degree" of $[1]$,

$$\langle\mu_{\mathrm{ev}},[1]\rangle := \tau^1(1) = \mathrm{Tr}\big[\psi_+^{(2n)}(1) - \psi_-^{(2n)}(1)\big] = \mathrm{Tr}(1-1) = 0.$$

The more complicated pairing is,

$$\langle\mu_{\mathrm{ev}}, e_{(2n)}\rangle := \tau^1(\mathrm{ch}_0\,e_{(2n)})$$
$$= \mathrm{Tr}\circ\psi_+^{(2n)}\circ(1-\sigma)\left(2^{n-1} + \tfrac{1}{2}[q^{-1}-1]^n x_0\right)$$
$$= (q^{-1}-1)^n\,\mathrm{Tr}[\psi_+^{(2n)}(x_0)] = (q^{-1}-1)^n(1-q^{-1})^{-n}$$
$$= (-1)^n.$$

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The fact that the matrix of pairings,

	$[1]$	$[e_{(2n)}]$
$[\varepsilon_\lambda]$	1	2^{n-1}
$[\mu_{\mathrm{ev}}]$	0	$(-1)^n$

is invertible over the integers proves that the classes $[1], [e_{(2n)}]$, both elements of the group $K_0[\mathcal{C}(S_q^{2n})] \cong \mathbb{Z}^2$, and the classes $[\varepsilon_\lambda], [\mu_{\mathrm{ev}}]$, both in $K^0[\mathcal{C}(S_q^{2n})] \cong \mathbb{Z}^2$, are nonzero and that no one of them may be a multiple of another class; thus they are generators of the respective groups.

Classically, the "degree" of the left-handed spinor bundle is -1. So, the K-homology class which correctly generalizes the classical K-orientation class $[\mu] \in K_0(S^{2n})$ is actually $(-1)^{n+1}[\mu_{\mathrm{ev}}]$.

6.6 K-theory for odd spheres

Again, define $[1] \in K_0[\mathcal{C}(S_q^{2n+1})]$ as the equivalence class of $1 \in \mathcal{C}(S_q^{2n+1})$. The pairing with our element $[\varepsilon_\lambda] \in K^0[\mathcal{C}(S_q^{2n+1})]$ is again,

$$\langle \varepsilon_\lambda, [1] \rangle := \tau^0(1) = 1.$$

There is no other independent generator in $K_0[\mathcal{C}(S_q^{2n+1})] \cong \mathbb{Z}$.

Instead, $K_1[\mathcal{C}(S_q^{2n+1})] \cong \mathbb{Z}$ is nonzero. So we need to construct a generator there. An odd K-theory element is an equivalence class of unitary matrices over the algebra. We can construct an appropriate sequence of unitary matrices recursively, just as we constructed the unipotents and idempotents.

Let $V_{(2n+1)} \in \mathrm{Mat}_{2^n}(A(S_q^{2n+1}))$ be defined recursively by

$$V_{(2n+1)} = \begin{pmatrix} x_{n+1} & q^{-1}V_{(2n-1)} \\ -V_{(2n-1)}^* & x_{n+1}^* \end{pmatrix}, \tag{6.13}$$

with $V_{(1)} = x_1$. By using the defining relations (5.3) one directly proves that it is unitary:

$$V_{(2n+1)}V_{(2n+1)}^* = V_{(2n+1)}^*V_{(2n+1)} = 1 . \tag{6.14}$$

In order to pair our K-homology element $[\mu_{\mathrm{odd}}] \in K^1[\mathcal{C}(S_q^{2n+1})]$ with the unitary $V_{(2n+1)}$, we need the lower degree part $\mathrm{ch}_{\frac{1}{2}}(V_{(2n+1)}) \in HC_1[A(S_q^{2n+1})]$ of its Chern character. It is given by the cyclic cycle,

$$\mathrm{ch}_{\frac{1}{2}}(V_{(2n+1)}) := \tfrac{1}{2}\,\mathrm{tr}\left(V_{(2n+1)} \otimes V_{(2n+1)}^* - V_{(2n+1)}^* \otimes V_{(2n+1)}\right)$$
$$= \tfrac{1}{2}(q^{-2} - 1)^n(x_1 \otimes x_1^* - x_1^* \otimes x_1). \tag{6.15}$$

Now, compute the pairing,

$$\langle \mu_{\mathrm{odd}}, V_{(2n+1)} \rangle := \langle \varphi, \mathrm{ch}_{\frac{1}{2}}(V_{(2n+1)}) \rangle$$
$$= -(q^{-2} - 1)^n \varphi(x_1^*, x_1)$$
$$= -\tfrac{1}{2}(q^{-2} - 1)^n \,\mathrm{Tr}\left(\psi(x_1^*)[F, \psi(x_1)]_-\right)$$
$$= -\tfrac{1}{2}(q^{-2} - 1)^n 2(1 - q^{-2})^{-n}$$
$$= (-1)^{n+1}.$$

This proves that $[V_{(2n+1)}] \in K_1[\mathcal{C}(S_q^{2n+1})]$ and $[\mu_{\mathrm{odd}}] \in K^1[\mathcal{C}(S_q^{2n+1})]$ are nonzero and that neither may be a multiple of another class. Thus $[V_{(2n+1)}]$ and $[\mu_{\mathrm{odd}}]$ are indeed generators of these groups.

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